The Eilenberg-Moore bicategory over a relative pseudomonad

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Definition 1. Let $\mathbb{C}, \mathbb{D}$ be bicategories and $J : \mathbb{D} \to \mathbb{C}$ a pseudofunctor. A *relative pseudomonad* $(T, i, {}^*; \eta, \mu, \theta)$ along J comprises:

- for every object $X \in \mathbb{D}$ an object $TX \in \mathbb{D}$ and map $i_X : JX \to TX$ in $\mathbb{C}$, and
- a family of functors $(-)^*_{X,Y} : \mathbb{C}(JX, TY) \to \mathbb{C}(TX, TY)$ for $X, Y \in \mathbb{D}$,

along with three families of invertible 2-cells:

- $\eta_f : f \to f^* i$ for $f : JX \to TY$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $f : JX \to TY$, $g : JW \to TX$, and
- $\theta_X : i^*_X \to 1_{TX}$ for $X \in \mathbb{D}$,

such that the following two coherence diagrams commute:

(i) for $f : JX \to TY$, $g : JW \to TX$, $h : JV \to TW$,

$$\begin{array}{ccccc}
((f^*g)^*h)^* & & \xrightarrow{\mu_{f^*g,h}} & & (f^*g)^*h^* \\
\downarrow{\scriptstyle (\mu_{f,g}h)^*} & & & & \downarrow{\scriptstyle \mu_{f,g}h^*} \\
((f^*g^*)h)^* & & & & (f^*g^*)h^* \\
\downarrow{\scriptstyle \sim} & & & & \downarrow{\scriptstyle \sim} \\
(f^*(g^*h))^* & \xrightarrow[\mu_{f,g^*h}]{} & f^*(g^*h)^* & \xrightarrow[f^*\mu_{g,h}]{} & f^*(g^*h^*)
\end{array} \tag{1}$$

(ii) for $f : JX \to TY$,

$$\begin{array}{ccccc}
f^* & \xrightarrow{(\eta_f)^*} & (f^*i)^* & \xrightarrow{\mu_{f,i}} & f^*i^* \\
 & \searrow{\scriptstyle \sim} & & & \downarrow{\scriptstyle f^*\theta} \\
 & & & & f^*1
\end{array} \tag{2}$$

We construct the Eilenberg-Moore bicategory over T, comprising *pseudoalgebras* as objects, *algebra morphisms* and *algebra 2-cells*.

Definition 2. Let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. A *pseudoalgebra* $(A, {}^a; \tilde{a}, \hat{a})$ comprises:

- an object $A \in \mathbb{C}$,
- a family of functors $(-)^a_X : \mathbb{C}(JX, A) \to \mathbb{C}(TX, A)$ for $X \in \mathbb{D}$,

along with two families of invertible 2-cells

- $\tilde{a}_f : f \to f^a i$ for $f : JX \to A$,
- $\hat{a}_{f,g} : (f^a g)^a \to f^a g^*$ for $f : JX \to A$, $g : JW \to TX$,

such that the following two coherence diagrams commute:

(i) for $f : JX \to A$, $g : JW \to TX$, $h : JV \to TW$,

$$\begin{array}{ccccc}
((f^a g)^a h)^a & & \xrightarrow{\hat{a}_{f^a g,h}} & & (f^a g)^a h^* \\
{\scriptstyle (\hat{a}_{f,g} h)^a}\downarrow & & & & \downarrow{\scriptstyle \hat{a}_{f,g} h^*} \\
((f^a g^*) h)^a & & & & (f^a g^*) h^* \\
{\scriptstyle \sim}\downarrow & & & & \downarrow{\scriptstyle \sim} \\
(f^a (g^* h))^a & \xrightarrow[\hat{a}_{f,g^* h}]{} & f^a (g^* h)^* & \xrightarrow[f^a \mu_{g,h}]{} & f^a (g^* h^*)
\end{array} \tag{3}$$

(ii) for $f : JX \to A$,

$$\begin{array}{ccccc}
f^a & \xrightarrow{(\tilde{a}_f)^a} & (f^a i)^a & \xrightarrow{\hat{a}_{f,i}} & f^a i^* \\
 & \searrow^{\sim} & & & \downarrow{\scriptstyle f^a \theta} \\
 & & & & f^a 1
\end{array} \tag{4}$$

Definition 3. Let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$, and let $(A, {}^a)$, $(B, {}^b)$ be pseudoalgebras over T. A *lax morphism of algebras* $(f, \bar{f})$ between $(A, {}^a)$ and $(B, {}^b)$ comprises:

- a map $f : A \to B$ in $\mathbb{C}$, and
- a family of 2-cells $\bar{f}_g : (fg)^b \to fg^a$ for $g : JX \to A$,

such that the following two coherence diagrams commute:

- for $g : JX \to A$ and $h : JW \to TX$,

$$\begin{array}{ccccc}
((fg)^b h)^b & & \xrightarrow{\hat{b}_{fg,h}} & & (fg)^b h^* \\
{\scriptstyle (\bar{f}_g h)^b}\downarrow & & & & \downarrow{\scriptstyle \bar{f}_g h^*} \\
((fg^a) h)^b & & & & (fg^a) h^* \\
{\scriptstyle \sim}\downarrow & & & & \downarrow{\scriptstyle \sim} \\
(f(g^a h))^b & \xrightarrow[\bar{f}_{g^a h}]{} & f(g^a h)^a & \xrightarrow[f\hat{a}_{g,h}]{} & f(g^a h^*)
\end{array} \tag{5}$$

- for $g : JX \to A$,

$$\begin{array}{ccccc}
fg & \xrightarrow{\tilde{b}_{fg}} & (fg)^b i & \xrightarrow{\bar{f}_g i} & (fg^a) i \\
 & \searrow_{f\tilde{a}_g} & & & \downarrow{\scriptstyle \sim} \\
 & & & & f(g^a i)
\end{array} \tag{6}$$

If all the $\bar{f}_g$ are invertible, we say $(f, \bar{f})$ is a *pseudomorphism*, and if they are all identities, we say $(f, \bar{f})$ is a *strict morphism*.

Definition 4. Let T be a relative pseudomonad along $J : \mathbb{D} \to C$, let $(A, {}^a)$ and $(B, {}^b)$ be pseudoalgebras over T, and let $(f, \bar{f})$ and $(f', \bar{f}')$ be morphisms of algebras between $(A, {}^a)$ and $(B, {}^b)$. An *algebra 2-cell* $\alpha : (f, \bar{f}) \implies (f', \bar{f}')$ is a 2-cell $\alpha : f \implies f'$ such that the following diagram commutes:

- for $g : JX \to A$,

$$\begin{array}{ccc} (fg)^b & \xrightarrow{(\alpha g)^b} & (f'g)^b \\ {\scriptstyle \bar{f}_g}\downarrow & & \downarrow{\scriptstyle \bar{f}'_g} \\ fg^a & \xrightarrow[\alpha g^a]{} & f'g^a \end{array} \tag{7}$$

Proposition 1. *There is a bicategory Ps-T-Alg$_l$ whose objects are pseudoalgebras, whose 1-cells are lax morphisms of algebras, and whose 2-cells are algebra 2-cells.*

Proof. We first show that for any pseudoalgebras $(A, {}^a)$ and $(B, {}^b)$, we have a hom-category of lax morphisms Ps-T-Alg$_l((A, {}^a), (B, {}^b))$. Since algebra 2-cells are just 2-cells in the underlying bicategory satisfying a property, it suffices to show that 1_f is an algebra 2-cell:

$$\begin{array}{ccc} (fg)^b & \xrightarrow{(1_f g)^b} & (fg)^b \\ {\scriptstyle \bar{f}_g}\downarrow & & \downarrow{\scriptstyle \bar{f}_g} \\ fg^a & \xrightarrow[1_f g^a]{} & fg^a \end{array}$$

and that if α and β are algebra 2-cells, then so is $\beta\alpha$:

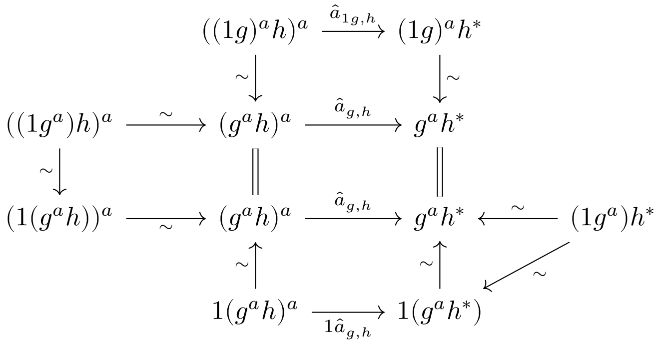
$$\begin{array}{ccccc} (fg)^b & \xrightarrow{(\alpha g)^b} & (f'g)^b & \xrightarrow{(\beta g)^b} & (f''g)^b \\ {\scriptstyle \bar{f}_g}\downarrow & & \downarrow{\scriptstyle \bar{f}'_g} & & \downarrow{\scriptstyle \bar{f}''_g} \\ fg^a & \xrightarrow[\alpha g^a]{} & f'g^a & \xrightarrow[\beta g^a]{} & f''g^a \end{array}$$

Hence indeed we have the required hom-categories.

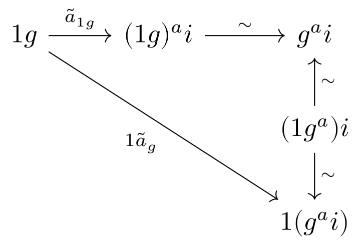
Next, we want identity functors $1_{(A,{}^a)} : \mathbb{1} \to$ Ps-T-Alg$_l((A, {}^a), (A, {}^a))$. We need to show that 1_A can be given the structure of a lax morphism of algebras. We equip it with the 2-cell $(\bar{1}_A)_g : (1_A g)^a \xrightarrow{\sim} g^a \xleftarrow{\sim} 1_A g^a$. The first coherence condition becomes

$$\begin{array}{ccccccc} ((1g)^a h)^a & & \xrightarrow{\hat{a}_{1g,h}} & & & & (1g)^a h^* \\ \sim\downarrow & & & & & & \downarrow\sim \\ (g^a h)^a & & & & & & g^a h^* \\ \sim\uparrow & & & & & & \uparrow\sim \\ ((1g^a)h)^a & & & & & & (1g^a)h^* \\ \sim\downarrow & & & & & & \downarrow\sim \\ (1(g^a h))^a & \xrightarrow[\sim]{} & (g^a h)^a & \xleftarrow[\sim]{} & 1(g^a h)^a & \xrightarrow[1\hat{a}_{g,h}]{} & 1(g^a h^*) \end{array}$$

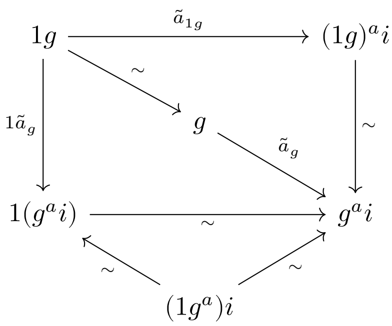
which we fill in as follows:



using two naturality squares and two associativity coherences from the underlying bicategory. The second coherence condition becomes



which we fill in as follows:



using two naturality squares and one associativity coherence from the underlying bicategory. Thus we can define our identity functor $1_{(A,{}^a)} : \mathbb{1} \to \text{Ps-}T\text{-Alg}_l((A,{}^a),(A,{}^a))$ as picking out the lax (in fact pseudo-) morphism $(1_A, \bar{1}_A)$.

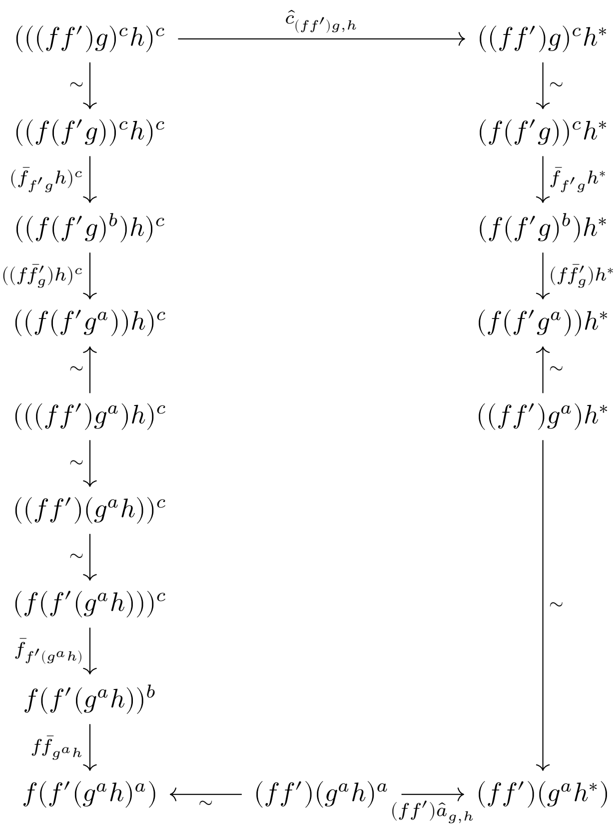
We also need to define horizontal composition: for every triple of objects $(A,{}^a)$, $(B,{}^b)$ and $(C,{}^c)$ a functor

$$\circ : \text{Ps-}T\text{-Alg}_l((B,{}^b),(C,{}^c)) \times \text{Ps-}T\text{-Alg}_l((A,{}^a),(B,{}^b)) \to \text{Ps-}T\text{-Alg}_l((A,{}^a),(C,{}^c)).$$

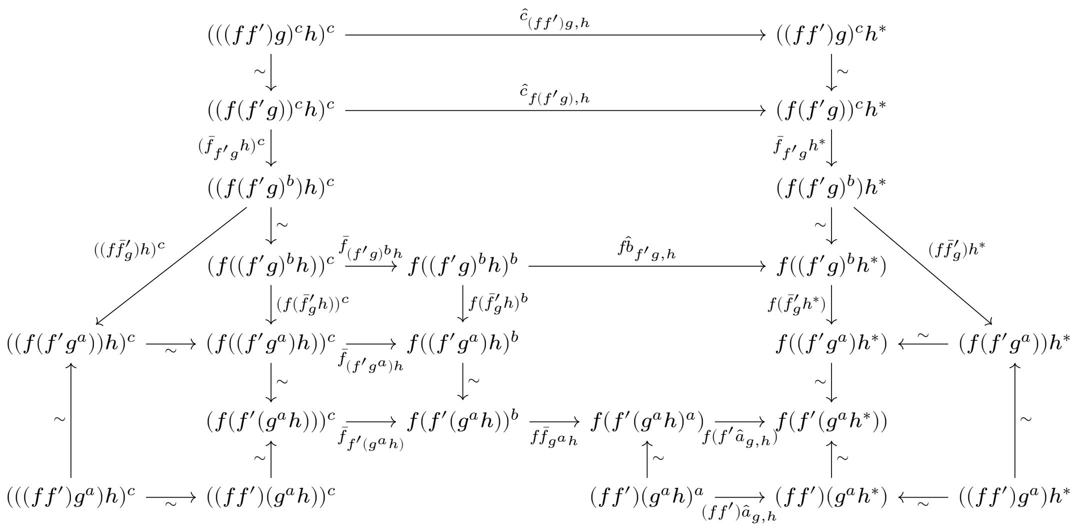
Given $(f,\bar{f})$ and $(f',\bar{f}')$ we define the horizontal composite $(f,\bar{f})\circ(f',\bar{f}')$ to be $(ff',\overline{ff'})$, where $\overline{ff'}_g$ for $g : JX \to A$ is defined as the composite

$$\overline{ff'}_g : ((ff')g)^c \xrightarrow{\sim} (f(f'g))^c \xrightarrow{\bar{f}_{f'g}} f(f'g)^b \xrightarrow{f\bar{f}'_g} f(f'g^a) \xleftarrow{\sim} (ff')g^a.$$

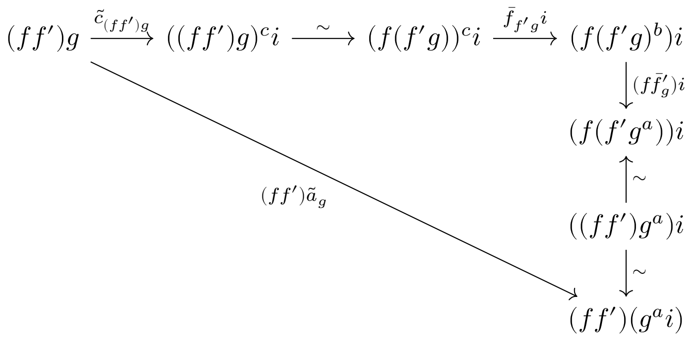
We must check that this composite actually gives ff' the structure of a lax morphism. Let $g : JX \to A$ and $h : JW \to TX$. The first coherence condition becomes



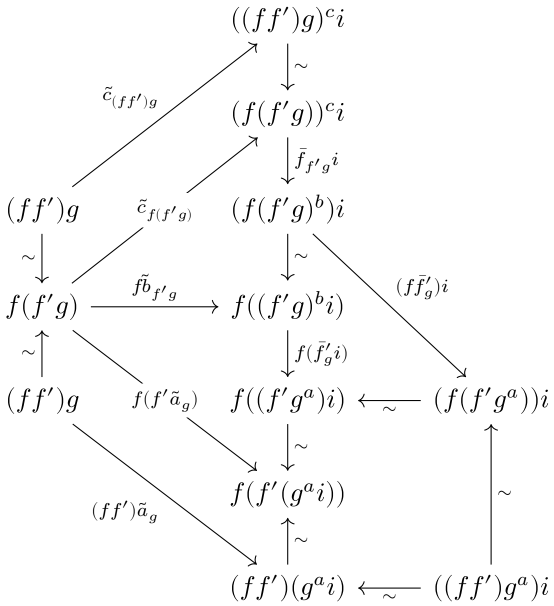
which we fill in as follows:



using six naturality squares, two pentagon axioms for the bicategory associator, and Equation 5 once each for f and f'. The second coherence condition becomes



which we can fill in as follows:



using three naturality squares, one pentagon axioms for the bicategory associator, and Equation 6 once each for f and f'. Hence $(ff', \overline{ff'})$ is a lax morphism and our horizontal composition functor is defined on 1-cells.

On 2-cells, since algebra 2-cells are just 2-cells satisfying a property, it suffices to check that the horizontal composite of algebra 2-cells is again an algebra 2-cell. So let $\alpha : (d, \bar{d}) \implies (f, \bar{f})$ and $\alpha' : (d', \bar{d}') \implies (f', \bar{f}')$ be algebra 2-cells. We need to verify the commutativity of

$$\begin{array}{ccc} ((dd')g)^c & \xrightarrow{((\alpha\cdot\alpha')g)^c} & ((ff')g)^c \\ {\scriptstyle \overline{dd'}_g}\downarrow & & \downarrow{\scriptstyle \overline{ff'}_g} \\ (dd')g^a & \xrightarrow[(\alpha\cdot\alpha')g^a]{} & (ff')g^a \end{array}$$

Using the definition of the lax morphism structures on dd' and ff', we can fill this diagram in as follows:

$$\begin{tikzcd}
((dd')g)^c \arrow[r, "((\alpha d')g)^c"] \arrow[d, "\sim"'] & ((fd')g)^c \arrow[r, "((f\alpha')g)^c"] \arrow[d, "\sim"'] & ((ff')g)^c \arrow[d, "\sim"] \\
(d(d'g))^c \arrow[r, "(\alpha(d'g))^c"] \arrow[d, "\bar{d}_{d'g}"'] & (f(d'g))^c \arrow[r, "(f(\alpha'g))^c"] \arrow[d, "\bar{f}_{d'g}"'] & (f(f'g))^c \arrow[d, "\bar{f}_{f'g}"] \\
d(d'g)^b \arrow[r, "\alpha(d'g)^b"] \arrow[d, "d\bar{d}'_g"'] & f(d'g)^b \arrow[r, "f(\alpha'g)^b"] \arrow[d, "f\bar{d}'_g"'] & f(f'g)^b \arrow[d, "f\bar{f}'_g"] \\
d(d'g^a) \arrow[r, "\alpha(d'g^a)"'] & f(d'g^a) \arrow[r, "f(\alpha'g^a)"'] & f(f'g^a) \\
(dd')g^a \arrow[r, "(\alpha d')g^a"'] \arrow[u, "\sim"] & (fd')g^a \arrow[r, "(f\alpha')g^a"'] \arrow[u, "\sim"] & (ff')g^a \arrow[u, "\sim"']
\end{tikzcd}$$

with six naturality squares and an instance of Equation 7 for each of α and α'. Thus horizontal composition is also defined on 2-cells. Furthermore, horizontal composition inherits functoriality from the underlying bicategory, since being an algebra 2-cell is a property.

We now define associator and unitor 2-cells by the same 2-cells in the underlying bicategory. We must check that these are algebra 2-cells. For the left unitor, we need

$$\begin{tikzcd}
((1f)g)^b \arrow[r, "\sim"] \arrow[d, "\overline{1f}_g"'] & (fg)^b \arrow[d, "\bar{f}_g"] \\
(1f)g^a \arrow[r, "\sim"'] & fg^a
\end{tikzcd}$$

and we can fill this in as follows:

$$\begin{tikzcd}
((1f)g)^b \arrow[d, "\sim"'] \arrow[ddr, "\sim"] & \\
(1(fg))^b \arrow[d, "\sim"'] & \\
(fg)^b \arrow[r, equal] & (fg)^b \arrow[dd, "\bar{f}_g"] \\
1(fg)^b \arrow[u, "\sim"] \arrow[d, "1\bar{f}_g"'] & \\
1(fg^a) \arrow[r, "\sim"] & fg^a \\
(1f)g^a \arrow[u, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

using a naturality square and two triangle identities for the underlying bicategory. Likewise, for the right unitor we need

$$\begin{array}{ccc} ((f1)g)^b & \xrightarrow{\sim} & (fg)^b \\ {\scriptstyle \overline{f1}_g}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}_g} \\ (f1)g^a & \xrightarrow[\sim]{} & fg^a \end{array}$$

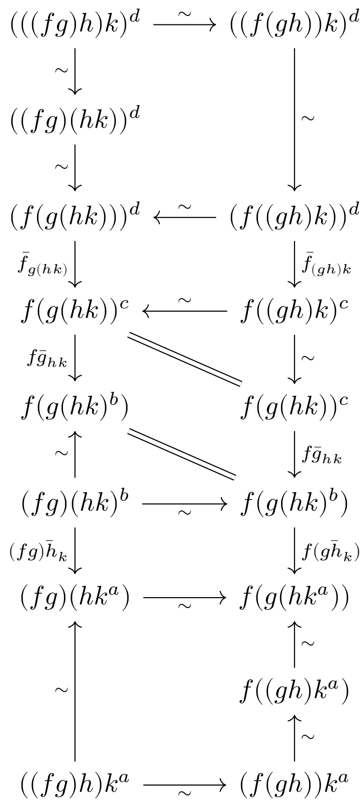
and we can fill this in as follows:

$$\begin{array}{ccc} ((f1)g)^b & & \\ {\scriptstyle \sim}\big\downarrow & \searrow^{\sim} & \\ (f(1g))^b & \xrightarrow[\sim]{} & (fg)^b \\ {\scriptstyle \bar{f}_{1g}}\big\downarrow & & \big| \\ f(1g)^a & & \big|{\scriptstyle \bar{f}_g} \\ {\scriptstyle \sim}\big\downarrow & & \big\downarrow \\ fg^a & = & fg^a \\ {\scriptstyle \sim}\big\uparrow & & \nearrow \\ f(1g^a) & \nearrow_{\sim} & \\ {\scriptstyle \sim}\big\uparrow & & \\ (f1)g^a & & \end{array}$$

using a naturality square and two triangle identities for the underlying bicategory. Finally, for the associator we need

$$\begin{array}{ccc} (((fg)h)k)^d & \xrightarrow{\sim} & ((f(gh))k)^d \\ {\scriptstyle \overline{(fg)h}_k}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f(gh)}_k} \\ ((fg)h)k^a & \xrightarrow[\sim]{} & (f(gh))k^a \end{array}$$

and we can fill this in as follows:



using two naturality squares and two pentagon axioms for the underlying bicategory. Hence the associator and unitors are algebra 2-cells; their pentagon and triangle axioms follow form them holding in the underlying bicategory.

Hence Ps-T-Alg$_l$ as defined is indeed a bicategory. □