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The 2-Category of Algebras over a Relative 2-Monad

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Introduction: the Presheaf Functor

Idea: We would like to give a monad structure to the presheaf construction

$$\mathbb{C} \mapsto \mathsf{Psh}\,\mathbb{C} := [\mathbb{C}^{\mathsf{op}}, \underline{\mathsf{Set}}].$$

We have an obvious choice of unit; for every (locally small) category there is the Yoneda embedding

$$\mathbb{C} \xrightarrow{Y} \mathsf{Psh}\,\mathbb{C} : A \mapsto \mathbb{C}(-, A).$$

The Presheaf Functor (cont.)

For the multiplication $M : \mathsf{Psh}\,\mathsf{Psh}\,\mathbb{C} \to \mathsf{Psh}\,\mathbb{C}$, we take the left Kan extension of $1_{\mathsf{Psh}\,\mathbb{C}}$ along the Yoneda embedding:

$$\begin{array}{ccc} \mathsf{Psh}\,\mathbb{C} & \xrightarrow{Y} & \mathsf{Psh}\,\mathsf{Psh}\,\mathbb{C} \\ & {\scriptstyle 1}\searrow \quad \Longrightarrow & \Big\downarrow {\scriptstyle M := \mathsf{Lan}_Y 1} \\ & & \mathsf{Psh}\,\mathbb{C} \end{array}$$

[This extension is guaranteed to exist, since $\mathsf{Psh}\,\mathbb{C}$ is cocomplete.] It is also worth noting the size issue here—we now need $\mathbb{C}$ to be small.

The Presheaf Functor (cont.)

We should not forget to check that Psh is a functor: it turns out that for an $F : \mathbb{C} \to \mathbb{D}$ we can define

$$\mathrm{Psh}(F) : \mathrm{Psh}\,\mathbb{C} \to \mathrm{Psh}\,\mathbb{D}$$

as the left adjoint of the pre-composition functor

$$[\mathbb{D}^{\mathrm{op}}, \underline{\mathrm{Set}}] \xrightarrow{-\circ F} [\mathbb{C}^{\mathrm{op}}, \underline{\mathrm{Set}}],$$

since $\underline{\mathrm{Set}}$ has all small colimits. (Somewhat more explicitly, $\mathrm{Psh}(F)$ is the left Kan extension of $Y \circ F : \mathbb{C} \to \mathbb{D} \to \mathrm{Psh}\,\mathbb{D}$ along the Yoneda embedding).

Now in 2D

Unfortunately, this is not enough to give us a genuine monad. The first problem is that none of the monad conditions hold on the nose: for example, the diagram

$$
\begin{array}{ccc}
\mathrm{Psh}\,\mathbb{C} & \xrightarrow{\;Y\;} & \mathrm{Psh}\,\mathrm{Psh}\,\mathbb{C} \\
 & \overset{1}{\searrow} \quad \Longrightarrow & \Big\downarrow{\scriptstyle M} \\
 & & \mathrm{Psh}\,\mathbb{C}
\end{array}
$$

commutes only up to the 2-cell shown.

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Size Issues

The second problem is more fundamental: Psh isn't really an endofunctor. For example, the category of presheaves on a small category is only locally-small.
Writing Cat for the 2-category of small categories, and CAT for the 2-category of locally-small categories, we have a 2-functor

$$\mathrm{Psh} : \mathrm{Cat} \to \mathrm{CAT}$$

To get around this issue, we will need to develop our definitions in the so-called ‘relative setting‘.

One-dimensional Ordinary Monads

Let (T, e, m) be a monad on a category $\mathbb{C}$. An algebra (A, a) over T consists of a carrier object $A \in \mathrm{ob}\,\mathbb{C}$ and algebra map $a : TA \to A$ making two diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{e} & TA \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad \begin{array}{ccc} T^2A & \xrightarrow{m} & TA \\ {\scriptstyle Ta}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array}$$

An algebra morphism $(A, a) \to (B, b)$ is given by an arrow $f : A \to B$ commuting with the algebra maps:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

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Relative Monads (Altenkirch, Chapman, Uustalu)

Let $\mathbb{D}, \mathbb{C}$ be categories and $J : \mathbb{D} \to \mathbb{C}$ a functor. A relative monad T along J consists of:

- For every $A \in \mathrm{ob}\,\mathbb{D}$ an object $TA \in \mathrm{ob}\,\mathbb{C}$ and map $e_A : JA \to TA$,
- For every $f : JA \to TB$ an extension $f^* : TA \to TB$, such that

$$(JA \xrightarrow{e} TA)^* = (TA \xrightarrow{1} TA),$$
$$(JA \xrightarrow{e} TA \xrightarrow{f^*} TB) = (JA \xrightarrow{f} TB)\ \forall f,$$
$$(TA \xrightarrow{g^*} TB \xrightarrow{f^*} TC) = (JA \xrightarrow{g} TB \xrightarrow{f^*} TC)^*\ \forall f, g.$$

These three equations correspond to the equations for an ordinary monad.

A relative monad along the identity $1_{\mathbb{C}}$ is a monad, and vice versa; in one direction, a relative monad $(T, e, {}^*)$ is given the structure of a functor via

$$(TA \xrightarrow{Tf} TB) := (A \xrightarrow{f} B \xrightarrow{e} TB)^*,$$

and multiplication given by

$$(T^2A \xrightarrow{m} TA) := (TA \xrightarrow{1} TA)^*.$$

In the other direction, given a monad (T, e, m) we can define the extension by

$$(A \xrightarrow{f} TB)^* := (TA \xrightarrow{Tf} T^2B \xrightarrow{m} TB).$$

It is worth noting that the three relative monad equations:

- $e_A^* = 1_{TA}$,
- $f^* \circ e_A = f$ for all $f : JA \to TB$,
- $(f^* \circ g)^* = f^* \circ g^*$ for all $g : JA \to TB$, $f : JB \to TC$

together *imply* that T is a functor $T : \mathbb{D} \to \mathbb{C}$, and that the collection of $e_A : JA \to TA$ forms a natural transformation $e : J \to T$.

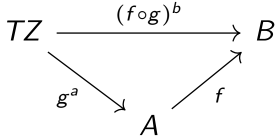
Algebras over a Relative Monad

Let $(T, e, {}^*)$ be a relative monad along $J : \mathbb{D} \to \mathbb{C}$. An <u>algebra</u> $(A, {}^a)$ over T consists of a <u>carrier object</u> $A \in \text{ob}\,\mathbb{C}$ and for every $f : JZ \to A$ a map $f^a : TZ \to A$, such that

$$(JZ \xrightarrow{e} TZ \xrightarrow{f^a} A) = (JZ \xrightarrow{f} A)\ \forall f,$$

$$(TY \xrightarrow{g^*} TZ \xrightarrow{f^a} A) = (JY \xrightarrow{g} TZ \xrightarrow{f^a} A)^a\ \forall f, g.$$

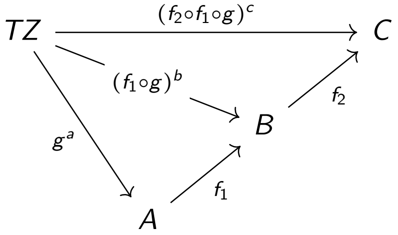
An <u>algebra morphism</u> $(A, {}^a) \to (B, {}^b)$ is given by a map $f : A \to B$ such that for all $g : JZ \to A$ we have that $f \circ g^a = (f \circ g)^b$:



An algebra over a relative monad along the identity $1_{\mathbb{C}}$ is equivalent to an algebra over an ordinary monad, and the morphisms also correspond. The equivalence is given by the definitions

$$(TA \xrightarrow{a} A) := (A \xrightarrow{1} A)^a,$$
$$(Z \xrightarrow{f} A)^a := (TZ \xrightarrow{Tf} TA \xrightarrow{a} A).$$

We still have a category of algebras: the composition of two algebra morphisms is again an algebra morphism in the relative setting:



1D Relative Setting: Summary

- Relative monad $(T, e, {}^*)$ along functor $J : \mathbb{D} \to \mathbb{C}$,
- Algebras $(A, {}^a)$ over T and algebra morphisms $(A, {}^a) \xrightarrow{f} (B, {}^b)$, forming
- Category of algebras T-Alg over a relative monad.

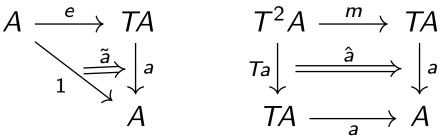
2D: Degrees of Strictness

- A strict 2-monad (T, e, m) consists of a strict 2-endofunctor T on a strict 2-category and strict 2-natural transformations e, m satisfying the monad laws strictly.
- A fully weak 2-monad (T, e, m) consists of a pseudofunctor T on a bicategory and pseudonatural transformations e, m satisfying the monad laws up to specified isomorphisms with coherence conditions.

Often we want to work somewhere in the middle: consider our example of Psh. The presheaf functor is between the strict 2-categories $\mathrm{Cat}, \mathrm{CAT}$ but the e, m we defined are not strict. For the sake of time, in this talk we will take our 2-monads to be strict.

Algebras over a 2-monad (Lack)

Let (T, e, m) be a 2-monad on a 2-category $\mathbb{C}$. A pseudoalgebra (A, a) over T consists of a carrier object $A \in \text{ob}\,\mathbb{C}$ and algebra map $a : TA \to A$ making two diagrams commute up to invertible 2-cells



called the unit and associativity of the pseudoalgebra, and these satisfy two coherence diagrams. If these 2-cells are in fact identities, we call (A, a) a strict algebra.

Morphisms of Algebras

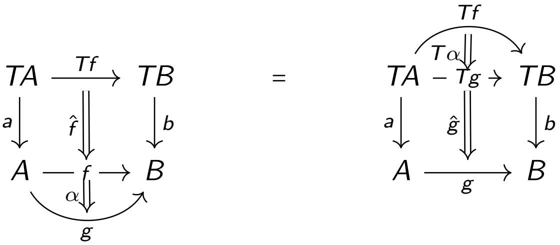
We have now three levels of strictness for algebra morphisms. A lax morphism of algebras $(A, a) \xrightarrow{f,\hat{f}} (B, b)$ consists of a morphism $A \xrightarrow{f} B$ and a 2-cell

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \overset{\hat{f}}{\Downarrow} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

satisfying two coherence conditions. If $\hat{f}$ is invertible we call $(f, \hat{f})$ a pseudomorphism, and if it is the identity we call $(f, \hat{f})$ a strict morphism of algebras.

Algebra 2-Cells

An <u>algebra 2-cell</u> $(f, \hat{f}) \xrightarrow{\alpha} (g, \hat{g})$ between algebra morphisms $(f, \hat{f}), (g, \hat{g}) : (A, a) \to (B, b)$ is a 2-cell $f \xrightarrow{\alpha} g$ for which



With algebras, algebra morphisms and algebra 2-cells we can form various 2-categories (strict when the underlying 2-category is strict).

Various 2-Categories

	Morphisms		
	Lax	Pseudo-	Strict
Strict algebras	T-Alg$_l$	T-Alg	T-Alg$_s$
Pseudoalgebras	Ps-T-Alg$_l$	Ps-T-Alg	Ps-T-Alg$_s$

Relative Pseudomonads (Fiore, Gambino, Hyland, Winskel)

A relative pseudomonad $(T, e, {}^*)$ along 2-functor $J : \mathbb{D} \to \mathbb{C}$ consists of, for every $A \in \operatorname{ob} \mathbb{D}$, an object $TA \in \operatorname{ob} \mathbb{C}$ and 1-cell $e_A : JA \to TA$, and for every A, B a *functor* $(-)^* : \mathbb{C}(JA, TB) \to \mathbb{C}(TA, TB)$, equipped with natural families of invertible 2-cells

$$(JA \xrightarrow{e} TA)^* \xRightarrow{\theta_A} (TA \xrightarrow{1} TA),$$

$$(JA \xrightarrow{e} TA \xrightarrow{f^*} TB) \xRightarrow{\eta_f} (JA \xrightarrow{f} TB) \ \forall f,$$

$$(TA \xrightarrow{g^*} TB \xrightarrow{f^*} TC) \xRightarrow{\mu_{f,g}} (JA \xrightarrow{g} TB \xrightarrow{f^*} TC)^* \ \forall f, g,$$

satisfying two coherence equations. For this talk, I will henceforth consider only (strict) relative 2-monads, where the θ, η, μ are all identities.

Algebras over Relative 2-monads

Let (T, e, m) be a relative 2-monad along $J : \mathbb{D} \to \mathbb{C}$. A pseudoalgebra (A, a) over T consists of a carrier object $A \in \text{ob}\,\mathbb{C}$ and functors $(-)^a : \mathbb{C}(JZ, A) \to \mathbb{C}(TZ, A)$, along with natural families of invertible 2-cells

$$(JZ \xrightarrow{f} A) \xRightarrow{\tilde{a}_f} (JZ \xrightarrow{e} TZ \xrightarrow{f^a} A) \;\forall f,$$

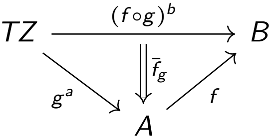
$$(JY \xrightarrow{g} TZ \xrightarrow{f^a} A)^a \xRightarrow{\hat{a}_{f,g}} (TY \xrightarrow{g^*} TZ \xrightarrow{f^a} A) \;\forall f, g,$$

satisfying two coherence diagrams. These 2-cells correspond to the two 2-cells in the definition of a pseudoalgebra over an ordinary 2-monad. If both these families of 2-cells are made up of identities, we have a strict algebra. For this talk, I will henceforth consider only strict algebras.

Algebra Morphisms

A lax morphism $(f, \bar{f})$ of algebras $(A, {}^a) \xrightarrow{(f,\bar{f})} (B, {}^b)$ is an arrow $A \xrightarrow{f} B$ together with a natural* family of 2-cells

$$(JZ \xrightarrow{g} A \xrightarrow{f} B)^b \xRightarrow{\bar{f}_g} (TZ \xrightarrow{g^a} A \xrightarrow{f} B) \ \forall g : JZ \to A,$$



satisfying two equalities of 2-cells (next slide). *Here 'naturality' means that for any 2-cell $g \xRightarrow{\alpha} g'$, we have

$$(f \cdot \alpha^a) \circ \bar{f}_g = \bar{f}_{g'} \circ (f \cdot \alpha)^b.$$

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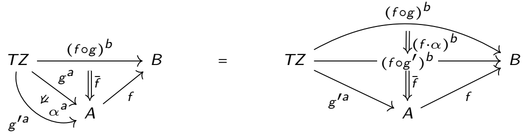
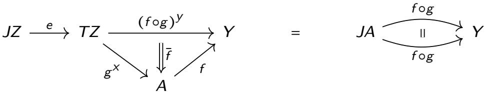
Diagrams of 2-Cells

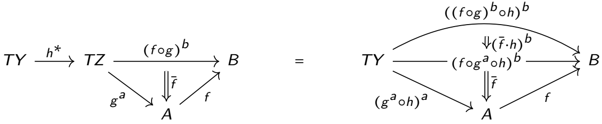
- Naturality: $(f \cdot \alpha^a) \circ \bar{f}_g = \bar{f}_{g'} \circ (f \cdot \alpha)^b$.



- Coherence with the unit: $\bar{f}_g \cdot e_Z = 1_{f \circ g}$ for every $g : JZ \to A$.



- Coherence with the extension: $\bar{f}_g \cdot h^* = \bar{f}_{g^a \circ h} \circ (\bar{f}_g \cdot h)^b$ for every $g : JZ \to A$, $h : JY \to TZ$.

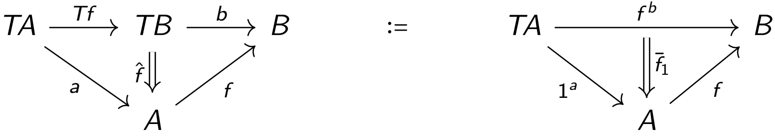


If all the $\bar{f}_g$ 2-cells are invertible, we call $(f, \bar{f})$ a pseudomorphism, and if they are identities, we call $(f, \bar{f})$ a strict morphism.

Proposition

A lax (pseudo-, strict) morphism of algebras over a relative 2-monad along the identity is equivalent to a lax (pseudo-, strict) morphism of algebras over an ordinary monad.

For example, since $b \circ Tf := f^b$ and $a := 1_A^a$, we have



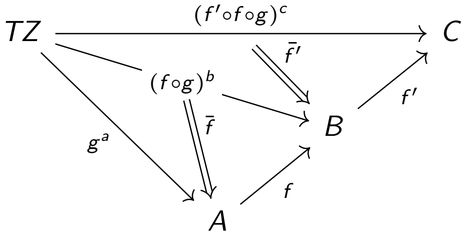
as part of the equivalence in one direction.

The Category of Algebras and Algebra Morphisms

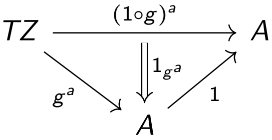
We can define the composite of two lax (pseudo-, strict) morphisms

$$(A, {}^{a}) \xrightarrow{(f,\bar{f})} (B, {}^{b}) \xrightarrow{(f',\bar{f}')} (C, {}^{c})$$

as $(f' \circ f, \overline{f' \circ f}) : (A, {}^{a}) \to (C, {}^{c})$ where $(\overline{f' \circ f})_g := \bar{f}'_{f \circ g} \circ (f' \cdot \bar{f}_h)$:



This composition is associative (by pasting another triangle on top) and we have identities for each $(A, {}^a)$ given by $(1_A, \overline{1_A})$ where $(\overline{1_A})_g := 1_{g^a}$.

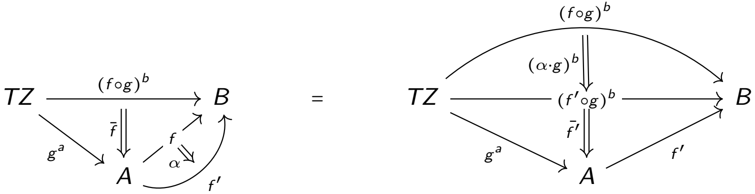


Hence we have a category of (strict) algebras over a relative 2-monad and lax (pseudo-, strict) morphisms between them.

Algebra 2-Cells

An algebra 2-cell $(f,\bar{f}) \xrightarrow{\alpha} (f,\bar{f}')$ between algebra morphisms $(f,\bar{f}),(f',\bar{f}') : (A,{}^a) \to (B,{}^b)$ is a 2-cell $f \xrightarrow{\alpha} f'$ for which

$$(\alpha \cdot g^a) \circ \bar{f} = \bar{f'} \circ (\alpha \cdot g)^b.$$



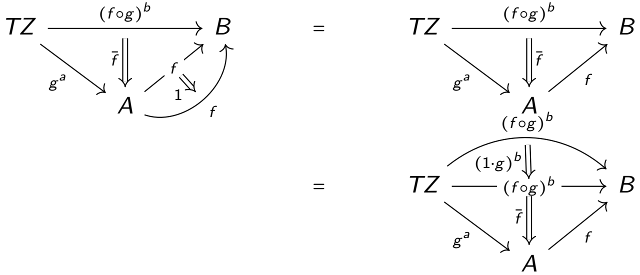
The 2-Category of Algebras over a Relative 2-Monad

Because the algebra 2-cells are just 2-cells from the underlying 2-category with an extra property, to show that "algebras, algebra morphisms and algebra 2-cells" form a 2-category we simply need to show the collection of algebra 2-cells is suitably closed (associativity, interchange, etc. follow from the structure of the underlying 2-category). We need to show:

- the identity 2-cell $(f, \bar{f}) \xRightarrow{1_f} (f, \bar{f})$ is an algebra 2-cell,
- the composite $\beta \circ \alpha$ of two algebra 2-cells $(f, \bar{f}) \xRightarrow{\alpha} (f_1, \bar{f}_1) \xRightarrow{\beta} (f_2, \bar{f}_2)$ is again an algebra 2-cell,
- the left whiskering $\alpha \cdot f : (f_1 \circ f, \overline{f_1 \circ f}) \implies (f_2 \circ f, \overline{f_2 \circ f})$ of an algebra 2-cell wih an algebra morphism is again an algebra 2-cell,
- the right whiskering $f \cdot \alpha : (f \circ f_1, \overline{f \circ f_1}) \implies (f \circ f_2, \overline{f \circ f_2})$ of an algebra 2-cell with an algebra morphism is again an algebra 2-cell.

Identities are Algebra 2-Cells

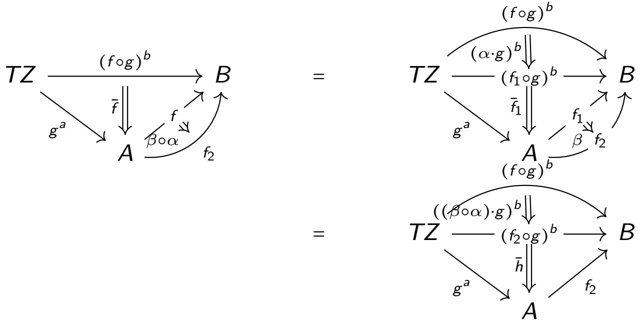
Since $(1_f \cdot g)^b = 1_{(f \circ g)^b}$,



and so indeed the identity is an algebra 2-cell.

Composites are Algebra 2-Cells

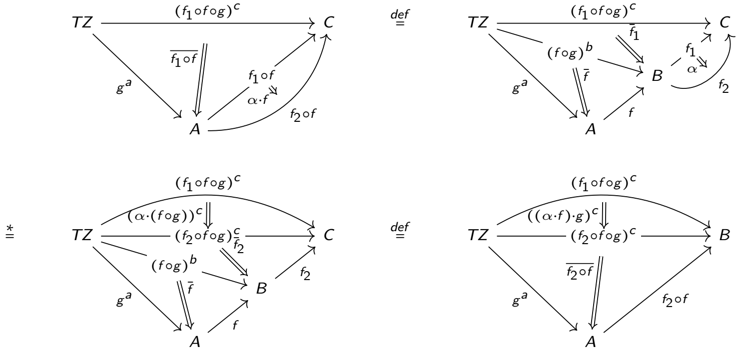
Since $((\beta \circ \alpha) \cdot g)^b = (\beta \cdot g)^b \circ (\alpha \cdot g)^b$,



and so indeed the composite of algebra 2-cells is again an algebra 2-cell.

Left Whiskerings are Algebra 2-Cells

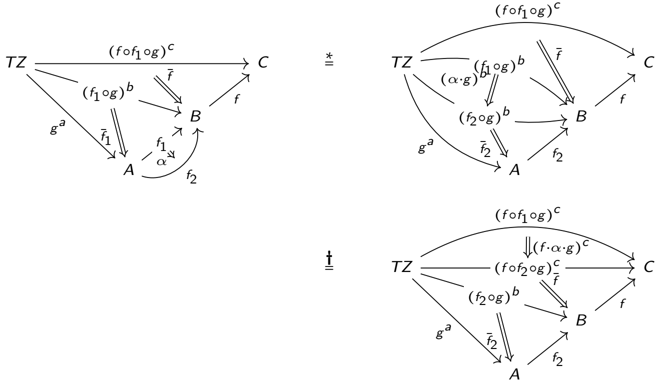
We have the series of equalities



(where the equality $*$ follows from α being an algebra 2-cell) and so indeed $\alpha \cdot f$ is an algebra 2-cell.

Right Whiskerings are Algebra 2-Cells

We have the equalities



(where $*$ follows from α being an algebra 2-cell, and † from the naturality condition) and so indeed right whiskerings are algebra 2-cells.

2-Categories of Algebras, Algebra Morphisms and Algebra 2-Cells

So we have verified that algebras over a relative 2-monad, lax (pseudo-, strict) morphisms and algebra 2-cells form a 2-category. We have 2-categories as shown:

	Morphisms		
	Lax	Pseudo-	Strict
Strict algebras	T-Alg_l	T-Alg	T-Alg_s
Pseudoalgebras			

Adding sufficiently-many invertible 2-cells to the previous slides in order to define Ps-T-Alg_l etc. is left as an exercise to the reader.

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References

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- Fiore, Gambino, Hyland, Winskel, 'Relative Pseudomonads, Kleisli Bicategories, and Substitution Monoidal Structures' (2017), arxiv.org/abs/1612.03678v3