RELATIVE MONADS ON SYMMETRIC MULTICATEGORIES

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We compare the definition of a strong relative monad between multicategories (defined by Slattery in [Sla23]) and of a strong relative monad between monoidal categories, as defined by Uustalu in [Uus10]).

Definition 0.1. A *relative monad* $(T, i, ^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each $A \in \mathrm{ob}\,\mathbb{C}$ an object TA and map $i_A : JA \to TA$, and
- for each $f : JA \to TB$ a map $f^* : TA \to TB$

such that we have

$$f = f^* i,$$
$$(f^* g)^* = f^* g^*,$$
$$i^* = 1$$

for all $g : JA \to TB$, $f : JB \to TC$.

$$\bullet \xrightarrow{\quad f \quad} \bullet \quad = \quad \bullet \xrightarrow{i} \bullet \xrightarrow{f^*} \bullet$$

$$(\bullet \xrightarrow{g} \bullet \xrightarrow{f^*} \bullet)^* \quad = \quad \bullet \xrightarrow{g^*} \bullet \xrightarrow{f^*} \bullet$$

$$\bullet \xrightarrow{\quad i^* \quad} \bullet \quad = \quad \bullet \overset{1}{=\!=\!=\!=} \bullet$$

T has the structure of a functor from $\mathbb{D}$ to $\mathbb{C}$, with action on maps given by

$$Tf := (if)^*.$$

$$T(\bullet \xrightarrow{\quad f \quad} \bullet) \quad = \quad (\bullet \xrightarrow{f} \bullet \xrightarrow{i} \bullet)^*$$

Indeed, a relative monad along the identity $1_{\mathbb{C}}$ is equivalent to an ordinary monad, with multiplication $m_X : TTX \to TX$ defined by

$$m_X := (1_{TX})^*.$$

In the rest of this document we abbreviate 'relative monad' to 'RM'.

1. Strength

Definition 1.1. A (multicategorical) *strong RM* $(T, i, ^t)$ along a multifunctor $J : \mathbb{D} \to \mathbb{C}$ between multicategories comprises

- for each $A \in \mathrm{ob}\,\mathbb{C}$ an object TA and map $i_A : JA \to TA$, and
- for each arity n, $1 \le j \le n$ and $f : A_1, ..., A_{j-1}, JX, A_{j+1}, ..., A_n \to TY$ a map $f^j : A_1, ..., TX, ..., A_n \to TY$, where $(-)^j$ is natural in all arguments,

Date: September 26, 2023.

such that we have

$$\begin{aligned} f &= f^j \circ_j i, \\ (f^j \circ_j g)^{j+k-1} &= f^j \circ_j g^k, \\ i^1 &= 1 \end{aligned}$$

for all $g : A_1, ..., JX, ..., A_m \to TY,\ f : B_1, ..., JY, ..., B_n \to TC$.

Note that such a strong RM is also a (unary) RM, with $(-)^*$ given by $(-)^1$.

Definition 1.2. A *J-strong RM* (T, i, t) along a lax monoidal functor $(J, \phi) : \mathbb{D} \to \mathbb{C}$ between monoidal categories comprises

- a relative monad $(T, i, ^*)$ along J, and
- a natural family of maps $t_{X,Y} : JX \times TY \to T(X \times Y)$

such that the following two diagrams commute:

$$\begin{array}{ccccc} I \times TX & \xrightarrow{\phi.\times 1} & JI \times TX & \xrightarrow{t} & T(I \times X) \\ & \searrow^{\lambda} & & {}^{T\lambda}\swarrow & \\ & & TX & & \end{array}$$

$$\begin{array}{ccccc} (JX \times JY) \times TZ & \xrightarrow{\phi\times 1} & J(X \times Y) \times TZ & \xrightarrow{t} & T((X \times Y) \times Z) \\ {\scriptstyle\alpha}\downarrow & & & & \downarrow{\scriptstyle T\alpha} \\ JX \times (JY \times TZ) & \xrightarrow[1\times t]{} & JX \times T(Y \times Z) & \xrightarrow[t]{} & T(X \times (Y \times Z)) \end{array}$$

(expressing coherence with the monoidal structure), and the following diagrams also commute:

$$\begin{array}{ccc} JX \times JY & \xrightarrow{\phi} & J(X \times Y) \\ {\scriptstyle 1\times i}\downarrow & & \downarrow{\scriptstyle i} \\ JX \times TY & \xrightarrow[t]{} & T(X \times Y) \end{array}$$

$$\begin{array}{ccc} JX \times JY & \xrightarrow{\phi} & J(X \times Y) \\ {\scriptstyle 1\times k}\downarrow & & {\scriptstyle l}\downarrow \\ JX \times TY' & \xrightarrow[t]{} & T(X \times Y') \end{array} \Longrightarrow \begin{array}{ccc} JX \times TY & \xrightarrow{t} & T(X \times Y) \\ \downarrow{\scriptstyle 1\times k^*} & & \downarrow{\scriptstyle l^*} \\ JX \times TY' & \xrightarrow[t]{} & T(X \times Y') \end{array}$$

(expressing coherence with the monad structure).

Given a representable multicategory $\mathbb{C}$, we have objects I, $X \times Y$ and maps $u : - \to I$, $\theta : X, Y \to X \times Y$ such that the induced maps

$$-\circ_j u : \mathbb{C}(..., I, ...; Y) \to \mathbb{C}(..., -, ...; Y),\ -\circ_j \theta : \mathbb{C}(..., X \times Y, ...; Z) \to \mathbb{C}(..., X, Y, ...; Z)$$

are isomorphisms. The maps λ, ρ, α may in this setting be defined by the equations

$$\begin{aligned} \lambda \circ \theta \circ_1 u &= 1, \\ \rho \circ \theta \circ_2 u &= 1, \\ \alpha \circ \theta \circ_1 \theta &= \theta \circ_2 \theta. \end{aligned}$$

Proposition 1.3. *Suppose that $\mathbb{D}$ and $\mathbb{C}$ are representable multicategories. Then a strong RM between these multicategories is a J-strong RM between $\mathbb{D}$ and C considered as monoidal categories, with strength map $t : JX \times TY \to T(X \times Y)$ defined by*

$$t \circ \theta := (i \circ J\theta)^2 : JX, TY \to J(X \times Y).$$

Proof. We have four properties to check. For the λ condition, we need

$$T\lambda \circ t \circ (\phi. \times 1) = \lambda.$$

We precompose with $\theta \circ_1 u$, a bijection on 1-cells, and obtain:

$$\begin{aligned} T\lambda \circ t \circ (\phi. \times 1) \circ \theta \circ_1 u &= T\lambda \circ t \circ \theta \circ_1 \phi. \circ_1 u \\ &= T\lambda \circ t \circ \theta \circ_1 Ju \\ &= T\lambda \circ (i \circ J\theta)^2 \circ_1 Ju \\ &= (i \circ J\lambda)^1 \circ (i \circ J\theta)^2 \circ_1 Ju \\ &= ((i \circ J\lambda)^1 \circ i \circ J\theta)^2 \circ_1 Ju \\ &= (i \circ J\lambda \circ J\theta)^2 \circ_1 Ju \\ &= (i \circ J\lambda \circ J\theta \circ_1 Ju)^1 \\ &= (i \circ J(\lambda \circ \theta \circ_1 u))^1 \\ &= (i \circ J1)^1 \\ &= i^1 = 1 \\ &= \lambda \circ \theta \circ_1 u, \end{aligned}$$

as required.

Next, for the α condition, we need

$$T\alpha \circ t \circ (\phi \times 1) = t \circ (1 \times t) \circ \alpha.$$

This time we precompose with $\theta \circ_1 \theta$ and obtain:

$$\begin{aligned}
T\alpha \circ t \circ (\phi \times 1) \circ \theta \circ_1 \theta &= T\alpha \circ t \circ \theta \circ_1 \phi \circ_1 \theta \\
&= T\alpha \circ t \circ \theta \circ_1 J\theta \\
&= T\alpha \circ (i \circ J\theta)^2 \circ_1 J\theta \\
&= (i \circ J\alpha)^1 \circ (i \circ J\theta)^2 \circ_1 J\theta \\
&= ((i \circ J\alpha)^1 \circ i \circ J\theta)^2 \circ_1 J\theta \\
&= (i \circ J\alpha \circ J\theta)^2 \circ_1 J\theta \\
&= (i \circ J\alpha \circ J\theta \circ_1 J\theta)^3 \\
&= (i \circ J(\alpha \circ \theta \circ_1 \theta))^3 \\
&= (i \circ J(\theta \circ_2 \theta))^3 \\
&= (i \circ J\theta \circ_2 J\theta)^3 \\
&= ((i \circ J\theta)^2 \circ_2 (i \circ J\theta))^3 \\
&= (i \circ J\theta)^2 \circ_2 (i \circ J\theta)^2 \\
&= t \circ \theta \circ_2 (t \circ \theta) \\
&= t \circ (1 \times t) \circ \theta \circ_2 \theta \\
&= t \circ (1 \times t) \circ \alpha \circ \theta \circ_1 \theta,
\end{aligned}$$

as required.

We move on to the two monad structure conditions. For the unit condition, we need

$$t \circ (1 \times i) = i \circ \phi.$$

Precomposing with θ, we obtain:

$$\begin{aligned}
t \circ (1 \times i) \circ \theta &= t \circ \theta \circ_2 i \\
&= (i \circ J\theta)^2 \circ_2 i \\
&= i \circ J\theta \\
&= icirc\phi \circ \theta,
\end{aligned}$$

as required.

Finally, for the extension condition, given

$$t \circ (1 \times k) = l \circ \phi,$$

we need to show that

$$t \circ (1 \times k^*) = l^* \circ t.$$

Precomposing with θ we obtain:

$$\begin{aligned}
t \circ (1 \times k^*) \circ \theta &= t \circ \theta \circ_2 k^1 \\
&= (i \circ J\theta)^2 \circ_2 k^1 \\
&= ((i \circ J\theta)^2 \circ_2 k)^2 \\
&= (t \circ (1 \times k^*) \circ \theta)^2 \\
&= (l \circ \phi \circ \theta)^2 \\
&= (l \circ J\theta)^2 \\
&= (l^1 \circ i \circ J\theta)^2 \\
&= l^1 \circ (i \circ J\theta)^2 \\
&= l^* \circ t,
\end{aligned}$$

as required. Hence a multicategorical RM between representable multicategories is a monoidal RM. □

References

[Sla23] Andrew Slattery. Pseudocommutativity and lax idempotency for relative pseudomonads, 2023.

[Uus10] Tarmo Uustalu. Strong Relative Monads (Extended Abstract), 2010. `https://cs.ioc.ee/~tarmo/papers/uustalu-cmcs10short.pdf`.

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