UNIVERSITY OF LEEDS

Pseudocommutativity for Relative Pseudomonads

Andrew Slattery

PSSL 107, 2nd April 2023

Outline

UNIVERSITY OF LEEDS

Outline

UNIVERSITY OF LEEDS

Outline

UNIVERSITY OF LEEDS

1. Background: the Presheaf Construction and Relative Pseudomonads
2. Background: Strong Monads and Commutative Monads
3. Parameterised Relative Pseudomonads and Multilinear Pseudofunctors
4. Pseudocommutative Relative Pseudomonads and Multilinear Pseudomonads
5. Coda: Lax idempotency

UNIVERSITY OF LEEDS

Outline

Outline

UNIVERSITY OF LEEDS

Outline

UNIVERSITY OF LEEDS

1. Background: the Presheaf Construction and Relative Pseudomonads
2. Background: Strong Monads and Commutative Monads
3. Parameterised Relative Pseudomonads and Multilinear Pseudofunctors
4. Pseudocommutative Relative Pseudomonads and Multilinear Pseudomonads
5. Coda: Lax idempotency

Extension systems

UNIVERSITY OF LEEDS

Definition

(Extension systems, Marmolejo + Wood 2010) An *extension system* $(T, i, ^*)$ on a category $\mathbb{C}$ comprises

- for each object A in $\mathbb{C}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : A \to TA$,
- for every map $f : A \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : A \to TB$, $g : Z \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A* = 1_{TA}.$$

Extension systems

Definition

(Extension systems, Marmolejo + Wood 2010) An *extension system* $(T, i, {}^*)$ on a category $\mathbb{C}$ comprises

- for each object A in $\mathbb{C}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : A \to TA$,
- for every map $f : A \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : A \to TB$, $g : Z \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A * = 1_{TA}.$$

Extension systems

UNIVERSITY OF LEEDS

Definition

(Extension systems, Marmolejo + Wood 2010) An *extension system* $(T, i, ^*)$ on a category $\mathbb{C}$ comprises

- for each object A in $\mathbb{C}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : A \to TA$,
- for every map $f : A \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : A \to TB$, $g : Z \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A * = 1_{TA}.$$

Extension systems

Definition

(Extension systems, Marmolejo + Wood 2010) An *extension system* $(T, i, ^*)$ on a category $\mathbb{C}$ comprises

- for each object A in $\mathbb{C}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : A \to TA$,
- for every map $f : A \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : A \to TB$, $g : Z \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A * = 1_{TA}.$$

Extension systems

Extension systems and monads are equivalent, in that each structure induces the other. However, the definition of extension system does not reference iteration of the action of T, and so it can be more easily generalised to the notion of a monad *along* some base functor $J : \mathbb{D} \to \mathbb{C}$.

Definition

(Relative monad, Altenkirch et al. 2014) A *relative monad* $(T, i, {}^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every map $f : JA \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : JA \to TB$, $g : JZ \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A * = 1_{TA}.$$

Extension systems

Extension systems and monads are equivalent, in that each structure induces the other. However, the definition of extension system does not reference iteration of the action of T, and so it can be more easily generalised to the notion of a monad *along* some base functor $J : \mathbb{D} \to \mathbb{C}$.

Definition

(Relative monad, Altenkirch et al. 2014) A *relative monad* $(T, i, {}^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every map $f : JA \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : JA \to TB$, $g : JZ \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A{*} = 1_{TA}.$$

Extension systems

UNIVERSITY OF LEEDS

Extension systems and monads are equivalent, in that each structure induces the other. However, the definition of extension system does not reference iteration of the action of T, and so it can be more easily generalised to the notion of a monad *along* some base functor $J : \mathbb{D} \to \mathbb{C}$.

Definition

(Relative monad, Altenkirch et al. 2014) A *relative monad* $(T, i, ^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every map $f : JA \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : JA \to TB$, $g : JZ \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A * = 1_{TA}.$$

Extension systems

Extension systems and monads are equivalent, in that each structure induces the other. However, the definition of extension system does not reference iteration of the action of T, and so it can be more easily generalised to the notion of a monad *along* some base functor $J : \mathbb{D} \to \mathbb{C}$.

Definition

(Relative monad, Altenkirch et al. 2014) A *relative monad* $(T, i, ^*)$ along a functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every map $f : JA \to TB$ an *extension* $f^* : TA \to TB$, satisfying the following three equations for all $f : JA \to TB$, $g : JZ \to TA$:

$$f = f^* i_A,$$
$$(f^* g)^* = f^* g^*,$$
$$i_A * = 1_{TA}.$$

UNIVERSITY OF LEEDS

Relative pseudomonads

We can categorify this definition, considering now 2-categories $\mathbb{C}$ and $\mathbb{D}$.

Definition

(Relative pseudomonad, Fiore et al. 2018) A *relative pseudomonad* $(T, i, ^*; \eta, \mu, \theta)$ along a 2-functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every A, B an extension functor between hom-categories $\mathbb{C}(JA, TB) \xrightarrow{(-)^*} \mathbb{C}(TA, TB)$,

along with three invertible families of 2-cells:

- $\eta_f : f \to f^* i_A$ for $f : JA \to TB$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $f : JA \to TB$, $g : JZ \to TA$, and
- $\theta_A : i_A^* \to 1_{TA}$ for A in $\mathbb{D}$.

satisfying two coherence diagrams.

Relative pseudomonads

We can categorify this definition, considering now 2-categories $\mathbb{C}$ and $\mathbb{D}$.

Definition

(Relative pseudomonad, Fiore et al. 2018) A *relative pseudomonad* $(T, i, ^*; \eta, \mu, \theta)$ along a 2-functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every A, B an extension functor between hom-categories $\mathbb{C}(JA, TB) \xrightarrow{(-)^*} \mathbb{C}(TA, TB)$,

along with three invertible families of 2-cells:

- $\eta_f : f \to f^* i_A$ for $f : JA \to TB$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $f : JA \to TB$, $g : JZ \to TA$, and
- $\theta_A : i_A^* \to 1_{TA}$ for A in $\mathbb{D}$.

satisfying two coherence diagrams.

Relative pseudomonads

We can categorify this definition, considering now 2-categories $\mathbb{C}$ and $\mathbb{D}$.

Definition

(Relative pseudomonad, Fiore et al. 2018) A *relative pseudomonad* $(T, i, ^*; \eta, \mu, \theta)$ along a 2-functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every A, B an extension functor between hom-categories $\mathbb{C}(JA, TB) \xrightarrow{(-)^*} \mathbb{C}(TA, TB)$,

along with three invertible families of 2-cells:

- $\eta_f : f \to f^* i_A$ for $f : JA \to TB$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $f : JA \to TB$, $g : JZ \to TA$, and
- $\theta_A : i_A^* \to 1_{TA}$ for A in $\mathbb{D}$.

satisfying two coherence diagrams.

Relative pseudomonads

UNIVERSITY OF LEEDS

We can categorify this definition, considering now 2-categories $\mathbb{C}$ and $\mathbb{D}$.

Definition

(Relative pseudomonad, Fiore et al. 2018) A *relative pseudomonad* $(T, i, {}^*; \eta, \mu, \theta)$ along a 2-functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every A, B an extension functor between hom-categories $\mathbb{C}(JA, TB) \xrightarrow{(-)^*} \mathbb{C}(TA, TB)$,

along with three invertible families of 2-cells:

- $\eta_f : f \to f^* i_A$ for $f : JA \to TB$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $f : JA \to TB$, $g : JZ \to TA$, and
- $\theta_A : i_A^* \to 1_{TA}$ for A in $\mathbb{D}$.

satisfying two coherence diagrams.

Relative pseudomonads

UNIVERSITY OF LEEDS

We can categorify this definition, considering now 2-categories $\mathbb{C}$ and $\mathbb{D}$.

Definition

(Relative pseudomonad, Fiore et al. 2018) A *relative pseudomonad* $(T, i, ^*; \eta, \mu, \theta)$ along a 2-functor $J : \mathbb{D} \to \mathbb{C}$ comprises

- for each object A in $\mathbb{D}$, an object TA in $\mathbb{C}$ and *unit map* $i_A : JA \to TA$,
- for every A, B an extension functor between hom-categories $\mathbb{C}(JA, TB) \xrightarrow{(-)^*} \mathbb{C}(TA, TB)$,

along with three invertible families of 2-cells:

- $\eta_f : f \to f^* i_A$ for $f : JA \to TB$,
- $\mu_{f,g} : (f^* g)^* \to f^* g^*$ for $f : JA \to TB$, $g : JZ \to TA$, and
- $\theta_A : i_A^* \to 1_{TA}$ for A in $\mathbb{D}$.

satisfying two coherence diagrams.

Example: The presheaf relative pseudomonad

The presheaf construction $X \mapsto \operatorname{Psh} X$ cannot be given the structure of a pseudomonad, since it is not an endofunctor (due to size issues). However, it can be given the structure of a relative pseudomonad along the inclusion $J : \mathrm{Cat} \to \mathrm{CAT}$ as follows:

- the unit $i_X : X \to \operatorname{Psh} X$ is given by the Yoneda embedding,
- the extension of a functor $f : X \to \operatorname{Psh} Y$ is given by the left Kan extension of f along the Yoneda embedding

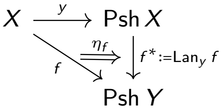
$$\begin{array}{ccc} X & \xrightarrow{\;y\;} & \operatorname{Psh} X \\ & \searrow^{f} \;\; \overset{\eta_f}{\Longrightarrow} & \big\downarrow f^* := \operatorname{Lan}_y f \\ & & \operatorname{Psh} Y \end{array}$$

  which also defines the 2-cells $\eta_f : f \to f^* i$.
- the 2-cells $\mu_{f,g}$ and θ_X are defined by the universal property of the left Kan extension.

Example: The presheaf relative pseudomonad

The presheaf construction $X \mapsto \mathrm{Psh}\, X$ cannot be given the structure of a pseudomonad, since it is not an endofunctor (due to size issues). However, it can be given the structure of a relative pseudomonad along the inclusion $J : \mathrm{Cat} \to \mathrm{CAT}$ as follows:

- the unit $i_X : X \to \mathrm{Psh}\, X$ is given by the Yoneda embedding,
- the extension of a functor $f : X \to \mathrm{Psh}\, Y$ is given by the left Kan extension of f along the Yoneda embedding

  which also defines the 2-cells $\eta_f : f \to f^* i$.
- the 2-cells $\mu_{f,g}$ and θ_X are defined by the universal property of the left Kan extension.

UNIVERSITY OF LEEDS

Strong monads and commutative monads

We briefly summarise the classical work of Anders Kock (1970) on monads on monoidal categories, which we wish to extend to relative pseudomonads on 2-multicategories.

- Every strong monad, being equipped with families of maps $t_{A,B} : A \otimes TB \to T(A \otimes B)$ and $s_{A,B} : TA \otimes B \to T(A \otimes B)$, is a lax monoidal functor with either of the structure maps

$$\phi_{A,B} : TA \otimes TB \xrightarrow{s} T(A \otimes TB) \xrightarrow{Tt} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B)$$

$$\phi'_{A,B} : TA \otimes TB \xrightarrow{t} T(TA \otimes B) \xrightarrow{Ts} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B).$$

- If the monad is furthermore *commutative* (meaning the two composites above are equal), then T is not only a monoidal functor but a monoidal monad.

UNIVERSITY OF LEEDS

Strong monads and commutative monads

We briefly summarise the classical work of Anders Kock (1970) on monads on monoidal categories, which we wish to extend to relative pseudomonads on 2-multicategories.

- Every strong monad, being equipped with families of maps $t_{A,B} : A \otimes TB \to T(A \otimes B)$ and $s_{A,B} : TA \otimes B \to T(A \otimes B)$, is a lax monoidal functor with either of the structure maps

$$\phi_{A,B} : TA \otimes TB \xrightarrow{s} T(A \otimes TB) \xrightarrow{Tt} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B)$$
$$\phi'_{A,B} : TA \otimes TB \xrightarrow{t} T(TA \otimes B) \xrightarrow{Ts} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B).$$

- If the monad is furthermore *commutative* (meaning the two composites above are equal), then T is not only a monoidal functor but a monoidal monad.

UNIVERSITY OF LEEDS

Strong monads and commutative monads

We briefly summarise the classical work of Anders Kock (1970) on monads on monoidal categories, which we wish to extend to relative pseudomonads on 2-multicategories.

- Every strong monad, being equipped with families of maps $t_{A,B} : A \otimes TB \to T(A \otimes B)$ and $s_{A,B} : TA \otimes B \to T(A \otimes B)$, is a lax monoidal functor with either of the structure maps

$$\phi_{A,B} : TA \otimes TB \xrightarrow{s} T(A \otimes TB) \xrightarrow{Tt} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B)$$

$$\phi'_{A,B} : TA \otimes TB \xrightarrow{t} T(TA \otimes B) \xrightarrow{Ts} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B).$$

- If the monad is furthermore *commutative* (meaning the two composites above are equal), then T is not only a monoidal functor but a monoidal monad.

UNIVERSITY OF LEEDS

Setting: 2-multicategories

Definition

(2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat.
Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \text{ob}\,\mathbb{C}$, together with
2. a category of multimorphisms $\mathbb{C}(X_1, \ldots, X_n; Y)$ for all $n \geq 0$ and objects $X_1, \ldots, X_n, Y$ which we call a *hom-category*,
3. an identity multimorphism functor $\mathbf{1}_X : 1 \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \text{ob}\,\mathbb{C}$, and
4. composition functors

$$\mathbb{C}(X_1, \ldots, X_n; Y) \times \mathbb{C}(W_{1,1}, \ldots, W_{1,m_1}) \times \ldots \times \mathbb{C}(W_{n,1}, \ldots, W_{n,m_n}) \to \mathbb{C}(W_{1,1}, \ldots, W_{n,m_n}; Y)$$
$$(f, g_1, \ldots, g_n) \mapsto f \circ (g_1, \ldots, g_n)$$

for all arities $n, m_1, \ldots, m_n$ and objects $Y, X_1, \ldots, X_n, W_{1,1}, \ldots, W_{n,m_n} \in \text{ob}\,\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

UNIVERSITY OF LEEDS

Setting: 2-multicategories

Definition

(2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat. Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \mathrm{ob}\,\mathbb{C}$, together with
2. a category of multimorphisms $\mathbb{C}(X_1, \ldots, X_n; Y)$ for all $n \geq 0$ and objects $X_1, \ldots, X_n, Y$ which we call a *hom-category*,
3. an identity multimorphism functor $\mathbf{1}_X : 1 \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \mathrm{ob}\,\mathbb{C}$, and
4. composition functors

$$\mathbb{C}(X_1, \ldots, X_n; Y) \times \mathbb{C}(W_{1,1}, \ldots, W_{1,m_1}) \times \ldots \times \mathbb{C}(W_{n,1}, \ldots, W_{n,m_n}) \to \mathbb{C}(W_{1,1}, \ldots, W_{n,m_n}; Y)$$
$$(f, g_1, \ldots, g_n) \mapsto f \circ (g_1, \ldots, g_n)$$

for all arities $n, m_1, \ldots, m_n$ and objects $Y, X_1, \ldots, X_n, W_{1,1}, \ldots, W_{n,m_n} \in \mathrm{ob}\,\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

Setting: 2-multicategories

UNIVERSITY OF LEEDS

Definition

(2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat. Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \mathrm{ob}\,\mathbb{C}$, together with
2. a category of multimorphisms $\mathbb{C}(X_1, \dots, X_n; Y)$ for all $n \geq 0$ and objects $X_1, \dots, X_n, Y$ which we call a *hom-category*,
3. an identity multimorphism functor $\mathbf{1}_X : 1 \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \mathrm{ob}\,\mathbb{C}$, and
4. composition functors

$$\mathbb{C}(X_1, \dots, X_n; Y) \times \mathbb{C}(W_{1,1}, \dots, W_{1,m_1}) \times \dots \times \mathbb{C}(W_{n,1}, \dots, W_{n,m_n}) \to \mathbb{C}(W_{1,1}, \dots, W_{n,m_n}; Y)$$
$$(f, g_1, \dots, g_n) \mapsto f \circ (g_1, \dots, g_n)$$

for all arities $n, m_1, \dots, m_n$ and objects $Y, X_1, \dots, X_n, W_{1,1}, \dots, W_{n,m_n} \in \mathrm{ob}\,\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

Setting: 2-multicategories

Definition

(2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat. Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \text{ob}\,\mathbb{C}$, together with
2. a category of multimorphisms $\mathbb{C}(X_1, \ldots, X_n; Y)$ for all $n \geq 0$ and objects $X_1, \ldots, X_n, Y$ which we call a *hom-category*,
3. an identity multimorphism functor $\mathbf{1}_X : 1 \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \text{ob}\,\mathbb{C}$, and
4. composition functors

$$\mathbb{C}(X_1, \ldots, X_n; Y) \times \mathbb{C}(W_{1,1}, \ldots, W_{1,m_1}) \times \ldots \times \mathbb{C}(W_{n,1}, \ldots, W_{n,m_n}) \to \mathbb{C}(W_{1,1}, \ldots, W_{n,m_n}; Y)$$
$$(f, g_1, \ldots, g_n) \mapsto f \circ (g_1, \ldots, g_n)$$

   for all arities $n, m_1, \ldots, m_n$ and objects $Y, X_1, \ldots, X_n, W_{1,1}, \ldots, W_{n,m_n} \in \text{ob}\,\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

UNIVERSITY OF LEEDS

Setting: 2-multicategories

Definition

(2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat. Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \operatorname{ob}\mathbb{C}$, together with
2. a category of multimorphisms $\mathbb{C}(X_1, \ldots, X_n; Y)$ for all $n \geq 0$ and objects $X_1, \ldots, X_n, Y$ which we call a *hom-category*,
3. an identity multimorphism functor $\mathbf{1}_X : 1 \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \operatorname{ob}\mathbb{C}$, and
4. composition functors

$$\mathbb{C}(X_1, \ldots, X_n; Y) \times \mathbb{C}(W_{1,1}, \ldots, W_{1,m_1}) \times \ldots \times \mathbb{C}(W_{n,1}, \ldots, W_{n,m_n}) \to \mathbb{C}(W_{1,1}, \ldots, W_{n,m_n}; Y)$$
$$(f, g_1, \ldots, g_n) \mapsto f \circ (g_1, \ldots, g_n)$$

for all arities $n, m_1, \ldots, m_n$ and objects $Y, X_1, \ldots, X_n, W_{1,1}, \ldots, W_{n,m_n} \in \operatorname{ob}\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

UNIVERSITY OF LEEDS

Setting: 2-multicategories

Definition

(2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat. Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \operatorname{ob}\mathbb{C}$, together with
2. a category of multimorphisms $\mathbb{C}(X_1, \ldots, X_n; Y)$ for all $n \geq 0$ and objects $X_1, \ldots, X_n, Y$ which we call a *hom-category*,
3. an identity multimorphism functor $\mathbf{1}_X : 1 \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \operatorname{ob}\mathbb{C}$, and
4. composition functors

$$\mathbb{C}(X_1, \ldots, X_n; Y) \times \mathbb{C}(W_{1,1}, \ldots, W_{1,m_1}) \times \ldots \times \mathbb{C}(W_{n,1}, \ldots, W_{n,m_n}) \to \mathbb{C}(W_{1,1}, \ldots, W_{n,m_n}; Y)$$
$$(f, g_1, \ldots, g_n) \mapsto f \circ (g_1, \ldots, g_n)$$

for all arities $n, m_1, \ldots, m_n$ and objects $Y, X_1, \ldots, X_n, W_{1,1}, \ldots, W_{n,m_n} \in \operatorname{ob}\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

Setting: 2-multicategories

UNIVERSITY OF LEEDS

We would like to relate 2-multicategories to perhaps more familiar structures.

Remark

- Every 2-multicategory restricts to a 2-category by considering only the unary hom-categories $\mathbb{C}(X; Y)$.
- Monoidal 2-categories (so in particular Cat and CAT) have underlying 2-multicategories, where hom-categories $\mathbb{C}(X_1, ..., X_n; Y)$ are given by $\mathbb{C}(X_1 \otimes ... \otimes X_n, Y)$ (choosing the leftmost bracketing of the tensor product).

To define a suitable notion of relative pseudomonad in the 2-multicategorical setting, we will extend a relative pseudomonad's unary extension functors $\mathbb{C}(JX, TY) \xrightarrow{(-)^*} \mathbb{C}(TX, TY)$ to general n-ary hom-categories.

UNIVERSITY OF LEEDS

Setting: 2-multicategories

We would like to relate 2-multicategories to perhaps more familiar structures.

Remark

- Every 2-multicategory restricts to a 2-category by considering only the unary hom-categories $\mathbb{C}(X; Y)$.
- Monoidal 2-categories (so in particular Cat and CAT) have underlying 2-multicategories, where hom-categories $\mathbb{C}(X_1, ..., X_n; Y)$ are given by $\mathbb{C}(X_1 \otimes ... \otimes X_n, Y)$ (choosing the leftmost bracketing of the tensor product).

To define a suitable notion of relative pseudomonad in the 2-multicategorical setting, we will extend a relative pseudomonad's unary extension functors $\mathbb{C}(JX, TY) \xrightarrow{(-)^*} \mathbb{C}(TX, TY)$ to general n-ary hom-categories.

Setting: 2-multicategories

UNIVERSITY OF LEEDS

We would like to relate 2-multicategories to perhaps more familiar structures.

Remark

- Every 2-multicategory restricts to a 2-category by considering only the unary hom-categories $\mathbb{C}(X; Y)$.
- Monoidal 2-categories (so in particular Cat and CAT) have underlying 2-multicategories, where hom-categories $\mathbb{C}(X_1, ..., X_n; Y)$ are given by $\mathbb{C}(X_1 \otimes ... \otimes X_n, Y)$ (choosing the leftmost bracketing of the tensor product).

To define a suitable notion of relative pseudomonad in the 2-multicategorical setting, we will extend a relative pseudomonad's unary extension functors $\mathbb{C}(JX, TY) \xrightarrow{(-)^*} \mathbb{C}(TX, TY)$ to general n-ary hom-categories.

Setting: 2-multicategories

We would like to relate 2-multicategories to perhaps more familiar structures.

Remark

- Every 2-multicategory restricts to a 2-category by considering only the unary hom-categories $\mathbb{C}(X; Y)$.
- Monoidal 2-categories (so in particular Cat and CAT) have underlying 2-multicategories, where hom-categories $\mathbb{C}(X_1, ..., X_n; Y)$ are given by $\mathbb{C}(X_1 \otimes ... \otimes X_n, Y)$ (choosing the leftmost bracketing of the tensor product).

To define a suitable notion of relative pseudomonad in the 2-multicategorical setting, we will extend a relative pseudomonad's unary extension functors $\mathbb{C}(JX, TY) \xrightarrow{(-)^*} \mathbb{C}(TX, TY)$ to general n-ary hom-categories.

UNIVERSITY OF LEEDS

Parameterised relative pseudomonads

Definition

(Parameterised relative pseudomonad) Let $\mathbb{C}$ and $\mathbb{D}$ be 2-multicategories and let $J : \mathbb{D} \to \mathbb{C}$ be a (unary) 2-functor between them. A *parameterised relative pseudomonad* $(T, i, {}^t; \tilde{t}, \hat{t}, \theta)$ along J comprises:

- for every object X in $\mathbb{D}$ an object TX in $\mathbb{C}$ and unit map $i_X : JX \to TX$,
- for every n, index $1 \leq i \leq n$, objects $B_1, ..., B_{i-1}, B_{i+1}, ..., B_n$ in $\mathbb{C}$ and objects X, Y in $\mathbb{D}$ a functor

$$\mathbb{C}(B_1, ..., B_{i-1}, JX, B_{i+1}, ..., B_n; TY) \xrightarrow{(-)^{t_i}} \mathbb{C}(B_1, ..., B_{i-1}, TX, B_{i+1}, ..., B_n; TY)$$

  called the *strength* (in the ith argument) and which is pseudonatural in all arguments, along with three natural families of invertible 2-cells:

- $\tilde{t}_f : f \to f^{t_j} \circ_j i$,
- $\hat{t}_{f,g} : (f^{t_j} \circ_j g)^{t_{j+k-1}} \to f^{t_j} \circ_j g^{t_k}$, and
- $\theta_X : (i_X)^{t_1} \to 1_{TX}$ for $f : B_1, ..., JX, ..., B_n \to TY$ and $g : C_1, ..., JW, ..., C_m \to TX$, satisfying two coherence diagrams.

UNIVERSITY OF LEEDS

Parameterised relative pseudomonads

For notational convenience, when a map such as $f : B_1, ..., JX, ..., B_n \to TY$ has only one explicitly possible strengthening index, we will denote this strengthening simply as f^t. We will furthermore use the notation $f^t \circ_t g$ to denote the composition of f^t with g in this strengthened argument. For example, the families of invertible 2-cells above are written in this notation as:

$$\tilde{t}_f : f \to f^t \circ_t i$$
$$\hat{t}_{f,g} : (f^t \circ_t g)^t \to f^t \circ_t g^t$$
$$\theta : i^t \to 1$$

The data for a parameterised relative pseudomonad resembles that for a (unary) relative pseudomonad very closely. Indeed, restricting $\mathbb{C}$ and $\mathbb{D}$ to their 2-categories of unary maps, $(T, i, ^t)$ is exactly a (unary) relative pseudomonad, with

$$(-)^* := (-)^t,$$
$$\eta := \tilde{t},$$
$$\mu := \hat{t},$$
$$\theta := \theta.$$

Parameterised relative pseudomonads

UNIVERSITY OF LEEDS

For notational convenience, when a map such as $f : B_1, ..., JX, ..., B_n \to TY$ has only one explicitly possible strengthening index, we will denote this strengthening simply as f^t. We will furthermore use the notation $f^t \circ_t g$ to denote the composition of f^t with g in this strengthened argument. For example, the families of invertible 2-cells above are written in this notation as:

$$\tilde{t}_f : f \to f^t \circ_t i$$
$$\hat{t}_{f,g} : (f^t \circ_t g)^t \to f^t \circ_t g^t$$
$$\theta : i^t \to 1$$

The data for a parameterised relative pseudomonad resembles that for a (unary) relative pseudomonad very closely. Indeed, restricting $\mathbb{C}$ and $\mathbb{D}$ to their 2-categories of unary maps, $(T, i, {}^t)$ is exactly a (unary) relative pseudomonad, with

$$(-)^* := (-)^t,$$
$$\eta := \tilde{t},$$
$$\mu := \hat{t},$$
$$\theta := \theta.$$

Parameterised relative pseudomonads

The stipulation that the maps

$$\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^{t_j}} \mathbb{C}(B_1, ..., TX, ..., B_n; TY)$$

be pseudonatural in all arguments asks in particular for invertible 2-cells of the form

- $(f \circ_k g)^{t_j} \cong f^{t_j} \circ_k g$ for $g : C_1, ..., C_m \to B_k$ (where $k \neq j$).

Wherever such pseudonaturality isomorphisms arise in diagrams we will leave them anonymous, as they can be inferred from the source and target.

UNIVERSITY OF LEEDS

Special case: strong monads

A strong monad structure on a monoidal category is given by a map $t_{X,Y} : X \otimes TY \to T(X \otimes Y)$ satisfying some axioms. To construct this map using a parameterised pseudomonad structure, we begin with the unit

$$i : X \otimes Y \to T(X \otimes Y).$$

Passing to the underlying multicategory, this corresponds to a map

$$i : X, Y \to T(X \otimes Y).$$

We can strengthen this map in the second argument to obtain

$$i^{t_2} : X, TY \to T(X \otimes Y).$$

Now passing back to the original monoidal category we have found a strength map $X \otimes TY \to T(X \otimes Y)$, and one can check that this satisfies the strength axioms. This derivation justifies the use of the terminology 'strength' to refer to the functors $\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^{t_i}} \mathbb{C}(B_1, ..., TX, ..., B_n; TY)$.

Special case: strong monads

UNIVERSITY OF LEEDS

A strong monad structure on a monoidal category is given by a map $t_{X,Y} : X \otimes TY \to T(X \otimes Y)$ satisfying some axioms. To construct this map using a parameterised pseudomonad structure, we begin with the unit

$$i : X \otimes Y \to T(X \otimes Y).$$

Passing to the underlying multicategory, this corresponds to a map

$$i : X, Y \to T(X \otimes Y).$$

We can strengthen this map in the second argument to obtain

$$i^{t_2} : X, TY \to T(X \otimes Y).$$

Now passing back to the original monoidal category we have found a strength map $X \otimes TY \to T(X \otimes Y)$, and one can check that this satisfies the strength axioms. This derivation justifies the use of the terminology 'strength' to refer to the functors $\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^{t_i}} \mathbb{C}(B_1, ..., TX, ..., B_n; TY)$.

Special case: strong monads

A strong monad structure on a monoidal category is given by a map $t_{X,Y} : X \otimes TY \to T(X \otimes Y)$ satisfying some axioms. To construct this map using a parameterised pseudomonad structure, we begin with the unit

$$i : X \otimes Y \to T(X \otimes Y).$$

Passing to the underlying multicategory, this corresponds to a map

$$i : X, Y \to T(X \otimes Y).$$

We can strengthen this map in the second argument to obtain

$$i^{t_2} : X, TY \to T(X \otimes Y).$$

Now passing back to the original monoidal category we have found a strength map $X \otimes TY \to T(X \otimes Y)$, and one can check that this satisfies the strength axioms. This derivation justifies the use of the terminology 'strength' to refer to the functors $\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^{t_i}} \mathbb{C}(B_1, ..., TX, ..., B_n; TY)$.

Special case: strong monads

A strong monad structure on a monoidal category is given by a map $t_{X,Y} : X \otimes TY \to T(X \otimes Y)$ satisfying some axioms. To construct this map using a parameterised pseudomonad structure, we begin with the unit

$$i : X \otimes Y \to T(X \otimes Y).$$

Passing to the underlying multicategory, this corresponds to a map

$$i : X, Y \to T(X \otimes Y).$$

We can strengthen this map in the second argument to obtain

$$i^{t_2} : X, TY \to T(X \otimes Y).$$

Now passing back to the original monoidal category we have found a strength map $X \otimes TY \to T(X \otimes Y)$, and one can check that this satisfies the strength axioms. This derivation justifies the use of the terminology 'strength' to refer to the functors $\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^{t_i}} \mathbb{C}(B_1, ..., TX, ..., B_n; TY)$.

UNIVERSITY OF LEEDS

Multilinear pseudofunctors

Definition

(Multilinear pseudofunctor) Given 2-multicategories $\mathbb{C}, \mathbb{D}$, a *multilinear pseudofunctor* $F : \mathbb{D} \to \mathbb{C}$ consists of:

- a function $\operatorname{ob}\mathbb{D} \xrightarrow{F} \operatorname{ob}\mathbb{C} : X \mapsto FX$,
- for each hom-category $\mathbb{D}(X_1, ..., X_n; Y)$ in $\mathbb{D}$ a functor

$$\mathbb{D}(X_1, ..., X_n; Y) \to \mathbb{C}(FX_1, ..., FX_n; FY) : f \mapsto Ff,$$

  along with
- for each $X \in \operatorname{ob}\mathbb{D}$ an invertible 2-cell

$$\tilde{F}_X : F1_X \implies 1_{FX},$$

- for each $f : X_1, ..., X_n \to Y$, $1 \leq i \leq n$ and $g : W_1, ..., W_m \to X_i$ an invertible 2-cell

$$\hat{F}_{f,g} : F(f \circ_i g) \implies Ff \circ_i Fg$$

satisfying three coherence conditions which parallel the unit and associativity diagrams for a lax monoidal functor.

UNIVERSITY OF LEEDS

Multilinear pseudofunctors

Definition

(Multilinear pseudofunctor) Given 2-multicategories $\mathbb{C}, \mathbb{D}$, a *multilinear pseudofunctor* $F : \mathbb{D} \to \mathbb{C}$ consists of:

- a function $\text{ob}\,\mathbb{D} \xrightarrow{F} \text{ob}\,\mathbb{C} : X \mapsto FX$,
- for each hom-category $\mathbb{D}(X_1, ..., X_n; Y)$ in $\mathbb{D}$ a functor

$$\mathbb{D}(X_1, ..., X_n; Y) \to \mathbb{C}(FX_1, ..., FX_n; FY) : f \mapsto Ff,$$

along with

- for each $X \in \text{ob}\,\mathbb{D}$ an invertible 2-cell

$$\tilde{F}_X : F1_X \implies 1_{FX},$$

- for each $f : X_1, ..., X_n \to Y$, $1 \le i \le n$ and $g : W_1, ..., W_m \to X_i$ an invertible 2-cell

$$\hat{F}_{f,g} : F(f \circ_i g) \implies Ff \circ_i Fg$$

satisfying three coherence conditions which parallel the unit and associativity diagrams for a lax monoidal functor.

Multilinear pseudofunctors

Definition

(Multilinear pseudofunctor) Given 2-multicategories $\mathbb{C}, \mathbb{D}$, a *multilinear pseudofunctor* $F : \mathbb{D} \to \mathbb{C}$ consists of:

- a function $\mathrm{ob}\,\mathbb{D} \xrightarrow{F} \mathrm{ob}\,\mathbb{C} : X \mapsto FX$,
- for each hom-category $\mathbb{D}(X_1, ..., X_n; Y)$ in $\mathbb{D}$ a functor

$$\mathbb{D}(X_1, ..., X_n; Y) \to \mathbb{C}(FX_1, ..., FX_n; FY) : f \mapsto Ff,$$

  along with
- for each $X \in \mathrm{ob}\,\mathbb{D}$ an invertible 2-cell

$$\tilde{F}_X : F1_X \implies 1_{FX},$$

- for each $f : X_1, ..., X_n \to Y$, $1 \le i \le n$ and $g : W_1, ..., W_m \to X_i$ an invertible 2-cell

$$\hat{F}_{f,g} : F(f \circ_i g) \implies Ff \circ_i Fg$$

satisfying three coherence conditions which parallel the unit and associativity diagrams for a lax monoidal functor.

Multilinear pseudofunctors

Definition

(Multilinear pseudofunctor) Given 2-multicategories $\mathbb{C}, \mathbb{D}$, a *multilinear pseudofunctor* $F : \mathbb{D} \to \mathbb{C}$ consists of:

- a function $\mathrm{ob}\,\mathbb{D} \xrightarrow{F} \mathrm{ob}\,\mathbb{C} : X \mapsto FX$,
- for each hom-category $\mathbb{D}(X_1, ..., X_n; Y)$ in $\mathbb{D}$ a functor

$$\mathbb{D}(X_1, ..., X_n; Y) \to \mathbb{C}(FX_1, ..., FX_n; FY) : f \mapsto Ff,$$

  along with

- for each $X \in \mathrm{ob}\,\mathbb{D}$ an invertible 2-cell

$$\tilde{F}_X : F1_X \implies 1_{FX},$$

- for each $f : X_1, ..., X_n \to Y$, $1 \leq i \leq n$ and $g : W_1, ..., W_m \to X_i$ an invertible 2-cell

$$\hat{F}_{f,g} : F(f \circ_i g) \implies Ff \circ_i Fg$$

satisfying three coherence conditions which parallel the unit and associativity diagrams for a lax monoidal functor.

Multilinear pseudofunctors

Just as every strong monad is lax monoidal as a functor, every parameterised relative pseudomonad is multilinear as a pseudofunctor.

Proposition

Let T be a parameterised relative pseudomonad along multilinear 2-functor $J : \mathbb{D} \to \mathbb{C}$. Then T is a multilinear pseudofunctor $T : \mathbb{D} \to \mathbb{C}$. The action of T on multimorphisms is given by

$$Tf := (i_Y \circ Jf)^{t_1 t_2 \cdots t_n} := \bar{f}^{t_1, \ldots, t_n}$$

for a map $f : X_1, \ldots, X_n \to Y$.

UNIVERSITY OF LEEDS

Multilinear pseudofunctors

Just as every strong monad is lax monoidal as a functor, every parameterised relative pseudomonad is multilinear as a pseudofunctor.

Proposition

Let T be a parameterised relative pseudomonad along multilinear 2-functor $J : \mathbb{D} \to \mathbb{C}$. Then T is a multilinear pseudofunctor $T : \mathbb{D} \to \mathbb{C}$. The action of T on multimorphisms is given by

$$Tf := (i_Y \circ Jf)^{t_1 t_2 \dots t_n} := \overline{f}^{t_1, \dots, t_n}$$

for a map $f : X_1, ..., X_n \to Y$.

UNIVERSITY OF LEEDS

Proof.

(modulo coherence) We need to construct invertible 2-cells $\tilde{T}_X : T1_X \implies 1_{TX}$ and $\hat{T}_{f,g} : T(f \circ_i g) \implies Tf \circ_i Tg$.
For the former, we can use the map

$$T1_X = (i_X \circ J1_X)^t = (i_X)^t \xrightarrow{\theta_X} 1_{TX}.$$

For the latter, we employ the composite

$$\begin{aligned} T(f \circ_i g) &= (i \circ Jf \circ_i Jg)^{t_1 \cdots t_{n+m-1}} \\ &\xrightarrow{\sim} ((i \circ Jf)^{t_1 \cdots t_{i-1}} \circ_i Jg)^{t_i \cdots t_{n+m-1}} \\ &\xrightarrow{\tilde{t}} ((i \circ Jf)^{t_1 \cdots t_i} \circ_i i \circ Jg)^{t_i \cdots t_{n+m-1}} \\ &\xrightarrow{\hat{t} \ldots \hat{t}} ((i \circ Jf)^{t_1 \cdots t_i} \circ_i (i \circ Jg)^{t_1 \cdots t_m})^{t_{i+m} \cdots t_{n+m-1}} \\ &\xrightarrow{\sim} (i \circ Jf)^{t_1 \cdots t_n} \circ_i (i \circ Jg)^{t_1 \cdots t_m} = Tf \circ_i Tg. \end{aligned}$$

□

Proof.

(modulo coherence) We need to construct invertible 2-cells $\tilde{T}_X : T1_X \Longrightarrow 1_{TX}$ and $\hat{T}_{f,g} : T(f \circ_i g) \Longrightarrow Tf \circ_i Tg$.
For the former, we can use the map

$$T1_X = (i_X \circ J1_X)^t = (i_X)^t \xrightarrow{\theta_X} 1_{TX}.$$

For the latter, we employ the composite

$$\begin{aligned} T(f \circ_i g) &= (i \circ Jf \circ_i Jg)^{t_1 \cdots t_{n+m-1}} \\ &\xrightarrow{\sim} ((i \circ Jf)^{t_1 \cdots t_{i-1}} \circ_i Jg)^{t_i \cdots t_{n+m-1}} \\ &\xrightarrow{\tilde{t}} ((i \circ Jf)^{t_1 \cdots t_i} \circ_i i \circ Jg)^{t_i \cdots t_{n+m-1}} \\ &\xrightarrow{\hat{t} \ldots \hat{t}} ((i \circ Jf)^{t_1 \cdots t_i} \circ_i (i \circ Jg)^{t_1 \cdots t_m})^{t_{i+m} \cdots t_{n+m-1}} \\ &\xrightarrow{\sim} (i \circ Jf)^{t_1 \cdots t_n} \circ_i (i \circ Jg)^{t_1 \cdots t_m} = Tf \circ_i Tg. \end{aligned}$$

□

UNIVERSITY OF LEEDS

Proof.

(modulo coherence) We need to construct invertible 2-cells $\tilde{T}_X : T1_X \Longrightarrow 1_{TX}$ and $\hat{T}_{f,g} : T(f \circ_i g) \Longrightarrow Tf \circ_i Tg$.
For the former, we can use the map

$$T1_X = (i_X \circ J1_X)^t = (i_X)^t \xrightarrow{\theta_X} 1_{TX}.$$

For the latter, we employ the composite

$$\begin{aligned} T(f \circ_i g) &= (i \circ Jf \circ_i Jg)^{t_1 \cdots t_{n+m-1}} \\ &\xrightarrow{\sim} ((i \circ Jf)^{t_1 \cdots t_{i-1}} \circ_i Jg)^{t_i \cdots t_{n+m-1}} \\ &\xrightarrow{\tilde{t}} ((i \circ Jf)^{t_1 \cdots t_i} \circ_i i \circ Jg)^{t_i \cdots t_{n+m-1}} \\ &\xrightarrow{\hat{t} \ldots \hat{t}} ((i \circ Jf)^{t_1 \cdots t_i} \circ_i (i \circ Jg)^{t_1 \cdots t_m})^{t_{i+m} \cdots t_{n+m-1}} \\ &\xrightarrow{\sim} (i \circ Jf)^{t_1 \cdots t_n} \circ_i (i \circ Jg)^{t_1 \cdots t_m} = Tf \circ_i Tg. \end{aligned}$$

□

Pseudocommutativity

There is some freedom in the multilinear pseudofunctorial structure we place on a given parameterised relative pseudomonad T; we defined the action of T on morphisms by

$$Tf := (i \circ Jf)^{t_1 \cdots t_n},$$

but we could equally well have chosen

$$Tf := (i \circ Jf)^{t_n \cdots t_1}$$

with the strengthenings applied in the reverse order. This parallels the classical situation described by Kock, where a strong monad with strength t and costrength s can be given the structure of lax monoidal functor in two ways:

$$TX \otimes TY \xrightarrow{t} T(TX \otimes Y) \xrightarrow{Ts} TT(X \otimes Y) \xrightarrow{\mu} T(X \otimes Y),$$
$$TX \otimes TY \xrightarrow{s} T(X \otimes TY) \xrightarrow{Tt} TT(X \otimes Y) \xrightarrow{\mu} T(X \otimes Y).$$

It is then natural to ask about those strong monads for which these two composites are equal, which Kock called *commutative monads*.

Pseudocommutativity

- Hyland and Power (2002) extend this notion to the two-dimensional setting, defining *pseudocommutativity* by asking only for an invertible 2-cell between the two composites. I will generalise further to the relative setting.
- Let us extend our notation in the following way. When a map $f : B_1, ..., JX, ..., JY, ..., B_n \to TZ$ has two explicitly possible strengthenings, let strengthening in the leftmost of these two arguments be denoted by f^s with 2-cells $\tilde{s} : f \to f^s \circ_s i$ and $\hat{s} : (f^s \circ_s g)^s \to f^s \circ g^t$, and let strengthening in the rightmost of these two arguments be denoted by f^t with 2-cells $\tilde{t}$, $\hat{t}$. When f has three explicitly possible strengthening we furthermore use f^u, etc.

Pseudocommutativity

- Hyland and Power (2002) extend this notion to the two-dimensional setting, defining *pseudocommutativity* by asking only for an invertible 2-cell between the two composites. I will generalise further to the relative setting.
- Let us extend our notation in the following way. When a map $f : B_1, \ldots, JX, \ldots, JY, \ldots, B_n \to TZ$ has two explicitly possible strengthenings, let strengthening in the leftmost of these two arguments be denoted by f^s with 2-cells $\tilde{s} : f \to f^s \circ_s i$ and $\hat{s} : (f^s \circ_s g)^s \to f^s \circ g^t$, and let strengthening in the rightmost of these two arguments be denoted by f^t with 2-cells $\tilde{t}$, $\hat{t}$. When f has three explicitly possible strengthening we furthermore use f^u, etc.

Pseudocommutativity

UNIVERSITY OF LEEDS

Definition

(Pseudocommutative relative pseudomonad) Let T be a parameterised relative pseudomonad. We call T *pseudocommutative* if for every pair of indices $1 \leq j < k \leq n$ and map

$$f : B_1, ..., B_{i-1}, JX, B_{i+1} ..., B_{j-1}, JY, B_{j+1}, ..., B_n \to TZ$$

we have an invertible 2-cell

$$\gamma_f : f^{ts} \to f^{st} : B_1, ..., TX, ..., TY, ..., B_n \to TZ$$

which is pseudonatural in all arguments and which satisfies five coherence conditions (one each for $\tilde{s}$, $\tilde{t}$, $\hat{s}$ and $\hat{t}$, along with a braiding condition).

The braiding condition

The braiding condition relates the six ways to strengthen a map

$$f : B_1, ...JW, ..., JX, ..., JY, ..., B_n \to TZ$$

in all three arguments, asking that the diagram

$$\begin{array}{ccccc} f^{uts} & \xrightarrow{(\gamma_f)^s} & f^{tus} & \xrightarrow{\gamma_{f^t}} & f^{tsu} \\ \downarrow{\scriptstyle \gamma_{f^u}} & & & & \downarrow{\scriptstyle (\gamma_f)^u} \\ f^{ust} & \xrightarrow[(\gamma_f)^t]{} & f^{sut} & \xrightarrow[\gamma_{f^s}]{} & f^{stu} \end{array}$$

commutes. Given a map $f : JX_1, ..., JX_n \to TY$ and a permutation $\sigma \in S_n$, we can construct maps

$$f^{t_1 \cdots t_n} \to f^{t_{\sigma(1)} \cdots t_{\sigma(n)}}$$

as a composite of γ maps and their inverses. The braiding axiom tells us that any two such composites of γ and γ^{-1} maps are equal.

The braiding condition

The braiding condition relates the six ways to strengthen a map

$$f : B_1, ...JW, ..., JX, ..., JY, ..., B_n \to TZ$$

in all three arguments, asking that the diagram

$$\begin{array}{ccccc} f^{uts} & \xrightarrow{(\gamma_f)^s} & f^{tus} & \xrightarrow{\gamma_{f^t}} & f^{tsu} \\ {\scriptstyle \gamma_{f^u}}\downarrow & & & & \downarrow{\scriptstyle (\gamma_f)^u} \\ f^{ust} & \xrightarrow[(\gamma_f)^t]{} & f^{sut} & \xrightarrow[\gamma_{f^s}]{} & f^{stu} \end{array}$$

commutes. Given a map $f : JX_1, ..., JX_n \to TY$ and a permutation $\sigma \in S_n$, we can construct maps

$$f^{t_1 \cdots t_n} \to f^{t_{\sigma(1)} \cdots t_{\sigma(n)}}$$

as a composite of γ maps and their inverses. The braiding axiom tells us that any two such composites of γ and γ^{-1} maps are equal.

Multilinear relative pseudomonads

UNIVERSITY OF LEEDS

Definition

Let $\mathbb{C}, \mathbb{D}$ be 2-multicategories and let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. We say T is a *multilinear relative pseudomonad* if

- T is a multilinear pseudofunctor, and
- The unit and extension of T are compatible with the pseudofunctor structure.

Explicitly, we ask that

- the monad unit i is multilinear: for each $f : X_1, \dots, X_n \to Y$ we have an invertible 2-cell

$$\bar{\imath}_f : i_Y \circ Jf \to Tf \circ (i_{X_1}, \dots, i_{X_n}),$$

$$\begin{array}{ccc} JX_1, \dots, JX_n & \xrightarrow{i, \dots, i} & TX_1, \dots, TX_n \\ {\scriptstyle Jf}\downarrow & \overset{\bar{\imath}_f}{\Nearrow} & \downarrow{\scriptstyle Tf} \\ JY & \xrightarrow[i]{} & TY \end{array}$$

Multilinear relative pseudomonads

- the monad extension $(-)^*$ is multilinear: for each 2-cell of the form $\alpha : h \circ Jf \to Tf' \circ (g_1, \ldots, g_n)$:

$$
\begin{array}{ccc}
JX_1, \ldots, JX_n & \xrightarrow{g_1, \ldots, g_n} & TX_1', \ldots, TX_n' \\
{\scriptstyle Jf}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle Tf'} \\
JY & \xrightarrow[h]{} & TY'
\end{array}
$$

we have a 2-cell $\alpha^* : h^* \circ Tf \to Tf' \circ (g_1^*, \ldots, g_n^*)$ fitting into the square

$$
\begin{array}{ccc}
TX_1, \ldots, TX_n & \xrightarrow{g_1^*, \ldots, g_n^*} & TX_1', \ldots, TX_n' \\
{\scriptstyle Tf}\downarrow & \overset{\alpha^*}{\nearrow} & \downarrow{\scriptstyle Tf'} \\
TY & \xrightarrow[h^*]{} & TY'
\end{array}
$$

and the $\bar{\imath}_f$ and the α^* satisfy three coherence conditions (one for each of the families of 2-cells making T a relative pseudomonad).

Multilinear relative pseudomonads

We shall see in the following theorem that for every parameterised relative pseudomonad, the monad unit is multilinear (we can define the invertible 2-cells $\bar{\imath}_f$), but in order to make the monad extension multilinear we require the relative pseudomonad to be pseudocommutative. This parallels the classical situation from Kock 1970, where the monad unit of a strong monad is always monoidal, but the monad multiplication is only monoidal if the monad is commutative.

Theorem

Let T be a parameterised relative pseudomonad along multilinear 2-functor $J : \mathbb{D} \to \mathbb{C}$. Suppose T is pseudocommutative. Then T is a multilinear relative pseudomonad.

Multilinear relative pseudomonads

We shall see in the following theorem that for every parameterised relative pseudomonad, the monad unit is multilinear (we can define the invertible 2-cells $\overline{\imath}_f$), but in order to make the monad extension multilinear we require the relative pseudomonad to be pseudocommutative. This parallels the classical situation from Kock 1970, where the monad unit of a strong monad is always monoidal, but the monad multiplication is only monoidal if the monad is commutative.

Theorem

Let T be a parameterised relative pseudomonad along multilinear 2-functor $J : \mathbb{D} \to \mathbb{C}$. Suppose T is pseudocommutative. Then T is a multilinear relative pseudomonad.

Proof (modulo coherence)

By the previous proposition, since T is parameterised we know that T is a multilinear pseudofunctor. We must check that the monad unit and extension are compatible with the pseudofunctor structure. We construct $\overline{\imath}_f$ as the composite

$$\begin{aligned} i \circ Jf &\xrightarrow{\tilde{t}} (i \circ Jf)^{t_1} \circ (i, 1, ..., 1) \\ &\xrightarrow{\tilde{t}} (i \circ Jf)^{t_1 t_2} \circ (i, i, 1, ..., 1) \\ &\vdots \\ &\xrightarrow{\tilde{t}} (i \circ Jf)^{t_1 t_2 \cdots t_n} \circ (i, i, i, ..., i) = Tf \circ (i, ..., i). \end{aligned}$$

Note that we do not need the pseudocommutativity to construct the $\overline{\imath}_f$ 2-cells.

Proof (modulo coherence)

By the previous proposition, since T is parameterised we know that T is a multilinear pseudofunctor. We must check that the monad unit and extension are compatible with the pseudofunctor structure. We construct $\bar{\imath}_f$ as the composite

$$
\begin{aligned}
i \circ Jf &\xrightarrow{\tilde{t}} (i \circ Jf)^{t_1} \circ (i, 1, ..., 1) \\
&\xrightarrow{\tilde{t}} (i \circ Jf)^{t_1 t_2} \circ (i, i, 1, ..., 1) \\
&\vdots \\
&\xrightarrow{\tilde{t}} (i \circ Jf)^{t_1 t_2 \cdots t_n} \circ (i, i, i, ..., i) = Tf \circ (i, ..., i).
\end{aligned}
$$

Note that we do not need the pseudocommutativity to construct the $\bar{\imath}_f$ 2-cells.

Proof (cont.)

UNIVERSITY OF LEEDS

The construction of α^* given $\alpha : h \circ Jf \to Tf' \circ (g_1, \ldots, g_n)$ is more involved. We require a 2-cell of shape

$$h^* \circ Tf \to Tf' \circ (g_1^*, \ldots, g_n^*).$$

We begin with the composite

$$\begin{aligned} h^* \circ Tf := h^t \circ (i \circ Jf)^{t_1 \cdots t_n} &\xrightarrow{\hat{t}^{-1}} (h^t \circ (i \circ Jf)^{t_1 \cdots t_{n-1}})^{t_n} \\ &\xrightarrow{\hat{t}^{-1}} (h^t \circ (i \circ Jf)^{t_1 \cdots t_{n-2}})^{t_{n-1} t_n} \\ &\vdots \\ &\xrightarrow{\hat{t}^{-1}} (h^t \circ i \circ Jf)^{t_1 \cdots t_n} \\ &\xrightarrow{\tilde{t}^{-1}} (h \circ Jf)^{t_1 \cdots t_n}, \end{aligned}$$

at which point we can compose with $\alpha^{t_1 \cdots t_n}$ to arrive at

$$(Tf' \circ (g_1, \ldots, g_n))^{t_1 \cdots t_n} := ((i \circ Jf')^{t_1 \cdots t_n} \circ (g_1, \ldots, g_n))^{t_1 \cdots t_n}.$$

Proof (cont.)

The construction of α^* given $\alpha : h \circ Jf \to Tf' \circ (g_1, ..., g_n)$ is more involved. We require a 2-cell of shape

$$h^* \circ Tf \to Tf' \circ (g_1^*, ..., g_n^*).$$

We begin with the composite

$$\begin{aligned} h^* \circ Tf := h^t \circ (i \circ Jf)^{t_1 \cdots t_n} &\xrightarrow{\hat{t}^{-1}} (h^t \circ (i \circ Jf)^{t_1 \cdots t_{n-1}})^{t_n} \\ &\xrightarrow{\hat{t}^{-1}} (h^t \circ (i \circ Jf)^{t_1 \cdots t_{n-2}})^{t_{n-1} t_n} \\ &\vdots \\ &\xrightarrow{\hat{t}^{-1}} (h^t \circ i \circ Jf)^{t_1 \cdots t_n} \\ &\xrightarrow{\tilde{t}^{-1}} (h \circ Jf)^{t_1 \cdots t_n}, \end{aligned}$$

at which point we can compose with $\alpha^{t_1 \cdots t_n}$ to arrive at

$$(Tf' \circ (g_1, ..., g_n))^{t_1 \cdots t_n} := ((i \circ Jf')^{t_1, \cdots, t_n} \circ (g_1, ..., g_n))^{t_1 \cdots t_n}.$$

Proof (cont.)

The construction of α^* given $\alpha : h \circ Jf \to Tf' \circ (g_1, ..., g_n)$ is more involved. We require a 2-cell of shape

$$h^* \circ Tf \to Tf' \circ (g_1^*, ..., g_n^*).$$

We begin with the composite

$$\begin{aligned} h^* \circ Tf := h^t \circ (i \circ Jf)^{t_1 \cdots t_n} &\xrightarrow{\hat{t}^{-1}} (h^t \circ (i \circ Jf)^{t_1 \cdots t_{n-1}})^{t_n} \\ &\xrightarrow{\hat{t}^{-1}} (h^t \circ (i \circ Jf)^{t_1 \cdots t_{n-2}})^{t_{n-1} t_n} \\ &\vdots \\ &\xrightarrow{\hat{t}^{-1}} (h^t \circ i \circ Jf)^{t_1 \cdots t_n} \\ &\xrightarrow{\tilde{t}^{-1}} (h \circ Jf)^{t_1 \cdots t_n}, \end{aligned}$$

at which point we can compose with $\alpha^{t_1 \cdots t_n}$ to arrive at

$$(Tf' \circ (g_1, ..., g_n))^{t_1 \cdots t_n} := ((i \circ Jf')^{t_1 \cdots t_n} \circ (g_1, ..., g_n))^{t_1 \cdots t_n}.$$

Proof (cont.)

From here we start needing the pseudocommutativity of T. Let $\sigma \in S_n$ be the cyclic permutation $1 \to 2 \to ... \to n \to 1$. Now we compose as follows:

$$
\begin{aligned}
((i \circ Jf')^{t_1 \cdot \ldots \cdot t_n} \circ (g_1, ..., g_n))^{t_1 \cdots t_n} &\xrightarrow{\gamma_\sigma} ((i \circ Jf')^{t_2 \cdot \ldots \cdot t_1} \circ (g_1, ..., g_n))^{t_1 \cdots t_n} \\
&\xrightarrow{\hat{t}} ((i \circ Jf')^{t_2 \cdot \ldots \cdot t_1} \circ (g_1^t, g_2, ..., g_n))^{t_2 \cdots t_n} \\
&\xrightarrow{\gamma_\sigma} ((i \circ Jf')^{t_3 \cdot \ldots \cdot t_2} \circ (g_1^t, g_2, ..., g_n))^{t_2 \cdots t_n} \\
&\xrightarrow{\hat{t}} ((i \circ Jf')^{t_3 \cdot \ldots \cdot t_2} \circ (g_1^t, g_2^t, g_3, ..., g_n))^{t_3 \cdots t_n} \\
&\vdots \\
&\xrightarrow{\gamma_\sigma} ((i \circ Jf')^{t_1 \cdot \ldots \cdot t_n} \circ (g_1^t, ..., g_{n-1}^t, g_t))^{t_n} \\
&\xrightarrow{\hat{t}} (i \circ Jf')^{t_1 \cdot \ldots \cdot t_n} \circ (g_1^t, ..., , g_n^t) \\
&= Tf' \circ (g_1^*, ..., g_n^*).
\end{aligned}
$$

Proof.

For example, the full composite in the case where f is a binary map is given by the diagram below:

$$
\begin{array}{ccccc}
h^t \circ (i \circ Jf)^{st} & & \overset{\alpha^*}{\dashrightarrow} & & f'^{st} \circ (g_1^t, g_2^t) \\
\downarrow \hat{t}^{-1} & & & & \uparrow \hat{t} \\
 & & & & (f'^{st} \circ (g_1^*, g_2))^t \\
 & & & & \uparrow \gamma \\
(h^t \circ (i \circ Jf)^s)^t & & & & (f'^{ts} \circ (g_1^*, g_2))^t \\
 & & & & \uparrow \hat{s} \\
\downarrow \hat{s}^{-1} & & & & (f'^{ts} \circ (g_1, g_2))^{st} \\
 & & & & \uparrow \gamma^{-1} \\
(h^t \circ i \circ Jf)^{st} & \xrightarrow[\tilde{t}^{-1}]{} & (h \circ Jf)^{st} & \xrightarrow[\alpha]{} & (f'^{st} \circ (g_1, g_2))^{st}
\end{array}
$$

□

Summary

UNIVERSITY OF LEEDS

The previous sections have proved the following implications for T a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$ between 2-multicategories:

- T parameterised $\implies$ T multilinear pseudofunctor, and
- T pseudocommutative $\implies$ T multilinear pseudomonad.

Working directly with pseudocommutativity and multilinearity can be tedious. In the coda we will examine a condition on a relative pseudomonad which both implies pseudocommutativity and which is much easier to verify, being characterised by a universal property.

Summary

The previous sections have proved the following implications for T a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$ between 2-multicategories:

- T parameterised $\implies$ T multilinear pseudofunctor, and
- T pseudocommutative $\implies$ T multilinear pseudomonad.

Working directly with pseudocommutativity and multilinearity can be tedious. In the coda we will examine a condition on a relative pseudomonad which both implies pseudocommutativity and which is much easier to verify, being characterised by a universal property.

Lax idempotency

The *lax-idempotent* relative pseudomonad generalises the notion of a lax-idempotent or Kock-Zöberlein 2-monad, discussed extensively in Kock (1995). The aim of this section is to generalise the result of Lopez Franco (2011) that every lax-idempotent 2-monad is pseudocommutative.

Definition

(Lax-idempotent relative pseudomonad, Fiore et al. 2018) Let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. We say that T is a *lax-idempotent relative pseudomonad* if 'monad structure is left adjoint to unit', which is to say that we have an adjunction

$$(-)^* : \mathbb{C}(JX, TY) \rightleftharpoons \mathbb{C}(TX, TY) : - \circ i$$

for all objects X, Y of $\mathbb{D}$, whose unit $- \implies (-)^* i$ has components given by the $\eta_f : f \to f^* i$ from the pseudomonadic structure (note in particular that the unit is thus invertible).

Lax idempotency

The *lax-idempotent* relative pseudomonad generalises the notion of a lax-idempotent or Kock-Zöberlein 2-monad, discussed extensively in Kock (1995). The aim of this section is to generalise the result of Lopez Franco (2011) that every lax-idempotent 2-monad is pseudocommutative.

Definition

(Lax-idempotent relative pseudomonad, Fiore et al. 2018) Let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. We say that T is a *lax-idempotent relative pseudomonad* if 'monad structure is left adjoint to unit', which is to say that we have an adjunction

$$(-)^* : \mathbb{C}(JX, TY) \rightleftharpoons \mathbb{C}(TX, TY) : - \circ i$$

for all objects X, Y of $\mathbb{D}$, whose unit $- \implies (-)^* i$ has components given by the $\eta_f : f \to f^* i$ from the pseudomonadic structure (note in particular that the unit is thus invertible).

UNIVERSITY OF LEEDS

Parameterised lax idempotency

Just as we defined the notion of a parameterised relative pseudomonad, we will adjust the definition of lax idempotency for the 2-multicategorical setting.

Definition

(Parameterised lax-idempotent relative pseudomonad) Let T be a parameterised relative pseudomonad along a multilinear pseudofunctor $J : \mathbb{D} \to \mathbb{C}$. We say T is a *parameterised lax-idempotent relative pseudomonad* if the strength is left adjoint to precomposition with the unit. That is, we have an adjunction

$$(-)^t : \mathbb{C}(B_1, ..., JX, ..., B_n; TY) \rightleftharpoons \mathbb{C}(B_1, ..., TX, ..., B_n; TY) : - \circ_t i$$

for $1 \leq j \leq n$ and objects $B_1, ..., JX, ..., B_n; TY$ whose unit $- \implies (-)^t \circ_t i$ has components

$$\tilde{t}_f : f \to f^t \circ_t i$$

obtained from the parameterised structure (again the unit is invertible).

Parameterised lax idempotency

Just as we defined the notion of a parameterised relative pseudomonad, we will adjust the definition of lax idempotency for the 2-multicategorical setting.

Definition

(Parameterised lax-idempotent relative pseudomonad) Let T be a parameterised relative pseudomonad along a multilinear pseudofunctor $J : \mathbb{D} \to \mathbb{C}$. We say T is a *parameterised lax-idempotent relative pseudomonad* if the strength is left adjoint to precomposition with the unit. That is, we have an adjunction

$$(-)^t : \mathbb{C}(B_1, ..., JX, ..., B_n; TY) \rightleftharpoons \mathbb{C}(B_1, ..., TX, ..., B_n; TY) : - \circ_t i$$

for $1 \leq j \leq n$ and objects $B_1, ..., JX, ..., B_n; TY$ whose unit $- \implies (-)^t \circ_t i$ has components

$$\tilde{t}_f : f \to f^t \circ_t i$$

obtained from the parameterised structure (again the unit is invertible).

Example: the presheaf relative pseudomonad

The parameterised structure for the presheaf relative pseudomonad is defined by the strengthening of a functor $f : B_1, ..., B_{j-1}, JX, B_{j+1}, ..., B_n \to \mathrm{Psh}\, Y$ being the left Kan extension

$$\begin{array}{ccc} B_1, ..., X, ..., B_n & \xrightarrow{1,\ldots,y,\ldots,1} & B_1, ..., \mathrm{Psh}\, X, ..., B_n \\ & \overset{\tilde{t}_f}{\Longrightarrow} & \Big\downarrow {\scriptstyle f^* := \mathrm{Lan}_{1,\ldots,y,\ldots,1} f} \\ & \searrow^{f} & \mathrm{Psh}\, Y \end{array}$$

along $1, ..., y, ..., 1$, and the 2-cell in the above diagram defines the map $\tilde{t}_f$. But this is exactly the statement that $(-)^t$ is left adjoint to $- \circ_t y$, and so Psh is a parameterised lax-idempotent relative pseudomonad.

Lax idempotent implies pseudocommutative

UNIVERSITY OF LEEDS

As a point of notation, we will use Greek letters to denote the *counit* of the lax idempotency adjunction; where the strengthening map is called $(-)^t$ and the unit $\tilde{t}$, the counit will be called

$$\tau_g : (g \circ_t i)^t \to g,$$

and where the strengthening is called $(-)^s$ and the unit $\tilde{s}$, the counit shall be called

$$\sigma_g : (g \circ_s i)^s \to g.$$

Theorem

Let $T : \mathbb{D} \to \mathbb{C}$ be a parameterised lax-idempotent relative pseudomonad. Then T is pseudocommutative, with a pseudocommutativity whose components $\gamma_g : g^{ts} \to g^{st}$ are given by the composite

$$g^{ts} \xrightarrow{(\tilde{s}_g)^{ts}} (g^s \circ_s i)^{ts} \xrightarrow{\sim} (g^{st} \circ_s i)^s \xrightarrow{\sigma_{g^{st}}} g^{st}.$$

Lax idempotent implies pseudocommutative

UNIVERSITY OF LEEDS

As a point of notation, we will use Greek letters to denote the *counit* of the lax idempotency adjunction; where the strengthening map is called $(-)^t$ and the unit $\tilde{t}$, the counit will be called

$$\tau_g : (g \circ_t i)^t \to g,$$

and where the strengthening is called $(-)^s$ and the unit $\tilde{s}$, the counit shall be called

$$\sigma_g : (g \circ_s i)^s \to g.$$

Theorem

Let $T : \mathbb{D} \to \mathbb{C}$ be a parameterised lax-idempotent relative pseudomonad. Then T is pseudocommutative, with a pseudocommutativity whose components $\gamma_g : g^{ts} \to g^{st}$ are given by the composite

$$g^{ts} \xrightarrow{(\tilde{s}_g)^{ts}} (g^s \circ_s i)^{ts} \xrightarrow{\sim} (g^{st} \circ_s i)^s \xrightarrow{\sigma_{g^{st}}} g^{st}.$$

Lax idempotent implies pseudocommutative

This theorem lets us deduce immediately that

Corollary

The relative pseudomonad Psh *is pseudocommutative.*

Combined with our earlier theorem, we find

Corollary

The relative pseudomonad Psh *is a multilinear relative pseudomonad.*

Lax idempotent implies pseudocommutative

This theorem lets us deduce immediately that

Corollary

The relative pseudomonad Psh *is pseudocommutative.*

Combined with our earlier theorem, we find

Corollary

The relative pseudomonad Psh *is a multilinear relative pseudomonad.*

Conclusion

We have the following implications:

- T parameterised $\implies$ T multilinear pseudofunctor,
- T pseudocommutative $\implies$ T multilinear pseudomonad, and
- T parameterised lax-idempotent $\implies$ T pseudocommutative.

From here I hope to explore:

- the effect of closure, symmetry and braiding conditions,
- when multilinear structure lifts to the Kleisli bicategory $\mathrm{Kl}(T)$ over a given relative pseudomonad.

Conclusion

We have the following implications:

- T parameterised $\implies$ T multilinear pseudofunctor,
- T pseudocommutative $\implies$ T multilinear pseudomonad, and
- T parameterised lax-idempotent $\implies$ T pseudocommutative.

From here I hope to explore:

- the effect of closure, symmetry and braiding conditions,
- when multilinear structure lifts to the Kleisli bicategory $\mathrm{Kl}(T)$ over a given relative pseudomonad.