# PSEUDOALGEBRAS FOR THE PRESHEAF RELATIVE PSEUDOMONAD ARE COCOMPLETE

ANDREW SLATTERY

## 1. Relative pseudomonads and their pseudoalgebras

**Definition 1.1.** Let $\mathbb{C}, \mathbb{D}$ be bicategories and $J : \mathbb{D} \to \mathbb{C}$ a pseudofunctor. A *relative pseudomonad* $(T, i, {}^*; \eta, \mu, \theta)$ along $J$ comprises:

- for every object $X \in \mathbb{D}$ an object $TX \in \mathbb{D}$ and map $i_X : JX \to TX$ in $\mathbb{C}$, and
- a family of functors $(-)^*_{X,Y} : \mathbb{C}(JX, TY) \to \mathbb{C}(TX, TY)$ for $X, Y \in \mathbb{D}$,

along with three families of invertible 2-cells:

- $\eta_f : f \to f^*i$ for $f : JX \to TY$,
- $\mu_{f,g} : (f^*g)^* \to f^*g^*$ for $f : JX \to TY$, $g : JW \to TX$, and
- $\theta_X : i^*_X \to 1_{TX}$ for $X \in \mathbb{D}$,

such that the following two coherence diagrams commute:

(i) for $f : JX \to TY$, $g : JW \to TX$, $h : JV \to TW$,

$$\begin{tikzcd}
((f^*g)^*h)^* \arrow[rr, "\mu_{f^*g,h}"] \arrow[d, "(\mu_{f,g}h)^*"'] & & (f^*g)^*h^* \arrow[d, "\mu_{f,g}h^*"] \\
((f^*g^*)h)^* \arrow[d, "\sim"'] & & (f^*g^*)h^* \arrow[d, "\sim"] \\
(f^*(g^*h))^* \arrow[r, "\mu_{f,g^*h}"'] & f^*(g^*h)^* \arrow[r, "f^*\mu_{g,h}"'] & f^*(g^*h^*)
\end{tikzcd} \tag{1}$$

(ii) for $f : JX \to TY$,

$$\begin{tikzcd}
f^* \arrow[r, "(\eta_f)^*"] \arrow[rrd, "\sim"'] & (f^*i)^* \arrow[r, "\mu_{f,i}"] & f^*i^* \arrow[d, "f^*\theta"] \\
& & f^*1
\end{tikzcd} \tag{2}$$

**Definition 1.2.** Let $T$ be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. A *pseudoalgebra* $(A, {}^a; \tilde{a}, \hat{a})$ comprises:

- an object $A \in \mathbb{C}$,
- a family of functors $(-)^a_X : \mathbb{C}(JX, A) \to \mathbb{C}(TX, A)$ for $X \in \mathbb{D}$,

along with two families of invertible 2-cells

- $\tilde{a}_f : f \to f^a i$ for $f : JX \to A$,
- $\hat{a}_{f,g} : (f^a g)^a \to f^a g^*$ for $f : JX \to A$, $g : JW \to TX$,

such that the following two coherence diagrams commute:

---

*Date*: September 29, 2023.

(i) for $f : JX \to A$, $g : JW \to TX$, $h : JV \to TW$,

$$\begin{tikzcd}
((f^a g)^a h)^a \arrow[rr, "\hat{a}_{f^a g, h}"] \arrow[d, "(\hat{a}_{f,g} h)^a"'] & & (f^a g)^a h^* \arrow[d, "\hat{a}_{f,g} h^*"] \\
((f^a g^*) h)^a \arrow[d, "\sim"'] & & (f^a g^*) h^* \arrow[d, "\sim"] \\
(f^a (g^* h))^a \arrow[r, "\hat{a}_{f, g^* h}"'] & f^a (g^* h)^* \arrow[r, "f^a \mu_{g,h}"'] & f^a (g^* h^*)
\end{tikzcd} \tag{3}$$

(ii) for $f : JX \to A$,

$$\begin{tikzcd}
f^a \arrow[r, "(\tilde{a}_f)^a"] \arrow[rrd, "\sim"'] & (f^a i)^a \arrow[r, "\hat{a}_{f,i}"] & f^a i^* \arrow[d, "f^a \theta"] \\
& & f^a 1
\end{tikzcd} \tag{4}$$

**Lemma 1.3.** *Let $\mathbb{A}$ be a pseudoalgebra over the relative pseudomonad $T$. Then for $f : JX \to A$, $g : JW \to TX$, the diagram*

$$\begin{tikzcd}
f^a g \arrow[r, "\tilde{a}"] \arrow[rrd, "f^a \eta"'] & (f^a g)^a i \arrow[r, "\hat{a}"] & (f^a g^*) i \arrow[d, "\sim"] \\
& & f^a (g^* i)
\end{tikzcd} \tag{5}$$

*also commutes.*

*Proof.* Since $\tilde{a}$ is a natural isomorphism, the required coherence is equivalent to

$$\begin{tikzcd}
(f^a g)^a i \arrow[r, "\tilde{a}"] \arrow[rrd, "\eta"'] & ((f^a g)^a i)^a i \arrow[r, "\hat{a}"] & ((f^a g^*) i)^a i \arrow[d, "\sim"] \\
& & (f^a (g^* i))^a i
\end{tikzcd}$$

and so it suffices to show

$$\begin{tikzcd}
(f^a g)^a \arrow[r, "\tilde{a}"] \arrow[rrd, "\eta"'] & ((f^a g)^a i)^a \arrow[r, "\hat{a}"] & ((f^a g^*) i)^a \arrow[d, "\sim"] \\
& & (f^a (g^* i))^a
\end{tikzcd}$$

commutes. Using this instance of equation (3)

$$\begin{tikzcd}
((f^a g)^a i)^a \arrow[r, "\hat{a}"] \arrow[dd, "\hat{a}"'] & ((f^a g^*) i)^a \arrow[r, "\sim"] & (f^a (g^* i))^a \arrow[d, "\hat{a}"] \\
& & f^a (g^* i)^* \arrow[d, "\mu"] \\
(f^a g)^a i^* \arrow[r, "\hat{a}"'] & (f^a g^*) i^* \arrow[r, "\sim"'] & f^a (g^* i^*)
\end{tikzcd}$$

the clockwise composite becomes

$$\begin{tikzcd}
(f^a g)^a \arrow[r, "\tilde{a}"] & ((f^a g)^a i)^a \arrow[r, "\hat{a}"] & (f^a g)^a i^* \arrow[r, "\hat{a}"] & (f^a g^*) i^* \arrow[d, "\sim"] \\
 & & & f^a(g^* i^*) \arrow[d, "\mu^{-1}"] \\
 & & & f^a(g^* i)^* \arrow[d, "\hat{a}^{-1}"] \\
 & & & (f^a(g^* i))^a
\end{tikzcd}$$

and now using equation (4) we can replace the first two maps in the clockwise composite to obtain:

$$\begin{tikzcd}
(f^a g)^a \arrow[r, "\sim"] & (f^a g)^a 1 \arrow[r, "\theta^{-1}"] & (f^a g)^a i^* \arrow[r, "\hat{a}"] & (f^a g^*) i^* \arrow[d, "\sim"] \\
 & & & f^a(g^* i^*) \arrow[d, "\mu^{-1}"] \\
 & & & f^a(g^* i)^* \arrow[d, "\hat{a}^{-1}"] \\
 & & & (f^a(g^* i))^a
\end{tikzcd}$$

which we fill in with four naturality squares, a bicategory coherences and relative pseudomonad coherence (2):

$$\begin{tikzcd}
(f^a g)^a \arrow[r, "\sim"] \arrow[d, "\hat{a}"'] & (f^a g)^a 1 \arrow[r, "\theta^{-1}"] \arrow[d, "\hat{a}"'] & (f^a g)^a i^* \arrow[d, "\hat{a}"] \\
f^a g^* \arrow[r, "\sim"] \arrow[d, equal] & (f^a g^*) 1 \arrow[r, "\theta^{-1}"] \arrow[d, "\sim"] & (f^a g^*) i^* \arrow[d, "\sim"] \\
f^a g^* \arrow[r, "\sim"] \arrow[dr, equal] & f^a(g^* 1) \arrow[r, "\theta^{-1}"] \arrow[d, "\sim"] & f^a(g^* i^*) \arrow[d, "\mu^{-1}"] \\
 & f^a g^* \arrow[r, "\eta"] \arrow[d, "\hat{a}^{-1}"'] & f^a(g^* i)^* \arrow[d, "\hat{a}^{-1}"] \\
 & (f^a g)^a \arrow[r, "\eta"'] & (f^a(g^* i))^a
\end{tikzcd}$$

Here the anticlockwise composite is equal to $(f^a \eta_g)^a$, as desired. ☐

## 2. The presheaf relative pseudomonad

Write Cat for the 2-category of small categories, and write CAT for the 2-category of locally-small categories. Since the category of presheaves on a small category is in general only locally small, it is natural to ask whether the presheaf construction

$$X \mapsto \operatorname{Psh} X := [X^{op}, \mathrm{Set}]$$

can be given the structure of a relative pseudomonad along the inclusion 2-functor $J : \mathrm{Cat} \to \mathrm{CAT}$.

This is shown in [FGHW18] via the construction of a relative pseudoadjunction; the structure of a relative pseudomonad is given to Psh as follows:

- for an object $X \in \mathrm{Cat}$ we have $\operatorname{Psh} X \in \mathrm{CAT}$ and unit map $y_X : X \to \operatorname{Psh} X$ given by the Yoneda embedding,
- for $X, Y \in \mathrm{Cat}$ and a functor $f : X \to \operatorname{Psh} Y$, the extension $f^* : \operatorname{Psh} X \to \operatorname{Psh} Y$ is given by the left Kan extension of $f$ along the Yoneda embedding

$$\begin{tikzcd}
X \arrow[r, "y"] \arrow[dr, "f"'] & \operatorname{Psh} X \arrow[d, "f^*:=\mathrm{Lan}_y f"] \arrow[d, phantom, shift right=20, "\overset{\eta_f}{\Longrightarrow}"] \\
& \operatorname{Psh} Y
\end{tikzcd}$$

  which also defines the 2-cells $\eta_f : f \to f^* y$ (note that since the Yoneda embedding is fully faithful the maps $\eta_f$ are invertible, as required),
- for $f : JX \to TY$ and $g : JW \to TX$, the 2-cell $\mu_{f,g} : (f^*g)^* \to f^*g^*$ is uniquely determined by the universal property of the left Kan extension:

$$\begin{tikzcd}
W \arrow[r, "y"] \arrow[dr, "g"'] & \operatorname{Psh} W \arrow[d, "g^*"] \arrow[d, phantom, shift right=12, "\overset{\eta_g}{\Longrightarrow}"] \\
& \operatorname{Psh} X \arrow[d, "f^*"] \\
& \operatorname{Psh} Y
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[rr, "y"] \arrow[ddrr, "f^*g"'] & & \operatorname{Psh} W \arrow[dd, "(f^*g)^*" description] \arrow[dr, "g^*"] & \\
& \overset{\eta_{f^*g}}{\Longrightarrow} & {} \arrow[r, phantom, "\overset{\mu_{f,g}}{\Longrightarrow}"] & \operatorname{Psh} X \arrow[dl, "f^*"] \\
& & \operatorname{Psh} Y &
\end{tikzcd}$$

- for $X \in \mathrm{Cat}$, the 2-cell $\theta_X : y_X^* \to 1$ is also uniquely determined by the universal property of the left Kan extension:

$$\begin{tikzcd}
X \arrow[r, "y"] \arrow[dr, "y"'] & \operatorname{Psh} X \arrow[d, "1"] \arrow[d, phantom, shift right=12, "\overset{1}{\Longrightarrow}"] \\
& \operatorname{Psh} X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "y"] \arrow[dr, "y"'] & \operatorname{Psh} X \arrow[d, dashed, "i^*" description] \arrow[d, phantom, shift right=12, "\overset{\eta_y}{\Longrightarrow}"] \arrow[d, phantom, shift left=12, "\overset{\theta_X}{\Longrightarrow}"] \arrow[d, bend left=60, "1"] \\
& \operatorname{Psh} X
\end{tikzcd}$$

The presheaf construction being the free cocompletion, we expect that pseudoalgebras over Psh should be exactly the locally-small cocomplete categories.

**Proposition 2.1.** *Let* $(\mathbb{A}, {}^a; \tilde{a}, \hat{a})$ *be a pseudoalgebra for the presheaf relative pseudomonad. Then* $\mathbb{A} \in \mathrm{CAT}$ *is cocomplete, and the colimit of the diagram* $F : \mathbb{D} \to \mathbb{A}$ *for* $\mathbb{D} \in \mathrm{Cat}$ *is given by*

$$\operatorname{colim} F \cong F^a \operatorname{colim} Y_{\mathbb{D}}.$$

*Proof.* Define $s := \operatorname{colim} Y_{\mathbb{D}} \in \operatorname{Psh} \mathbb{D}$ ($s$ standing for 'singleton', since it is the terminal presheaf on $\mathbb{D}$ sending every $d$ to a singleton set). Let $\mathbb{A}$ and $F : \mathbb{D} \to \mathbb{A}$ be as hypothesised. Our proof proceeds as follows:

1. Show that $F^a s$ is the apex of a cocone under $F$.
2. Given any cocone under $F$ with apex $Gt$, construct a map $z_G : F^a s \to Gt$.
3. Show that $z_G$ is a map of cocones.
4. Show that if $g : F^a s \to Gt$ is a map of cocones parallel to $z_G$, then $g = z_G$.

For (1), consider the colimit cocone under $Y_{\mathbb{D}}$ whose legs $\upsilon_d : Yd \to \operatorname{colim} Y$ are the colimit inclusions. Now for $d \in \mathbb{D}$ we can construct the composite

$$Fd \xrightarrow{\tilde{a}} F^aYd \xrightarrow{F^a\upsilon_d} F^a s.$$

Since the following diagram comprising a naturality square and the image under $F^a$ of a cocone:

$$\begin{tikzcd}
Fd \arrow[rr, "Ff"] \arrow[d, "\tilde{a}"'] & & Fd' \arrow[d, "\tilde{a}"] \\
F^aYd \arrow[rr, "F^aYf"] \arrow[dr, "F^a\upsilon_d"'] & & F^aYd' \arrow[dl, "F^a\upsilon_{d'}"] \\
& F^a s &
\end{tikzcd}$$

commutes for all $f : d \to d'$, these form the legs of the required cocone under $F$ with apex $F^a s$.

For (2), we begin by characterising cocones under $F$. Define $\mathbb{D}^t \in \mathrm{Cat}$ to be the category formed by freely adding a terminal object to $\mathbb{D}$; that is,

$$\operatorname{ob} \mathbb{D}^t = \operatorname{ob} \mathbb{D} \sqcup \{t\},$$

$$\operatorname{mor} \mathbb{D}^t = \operatorname{mor} \mathbb{D} \sqcup \{1_t\} \sqcup \{d \xrightarrow{!_d} t : d \in \operatorname{ob} \mathbb{D}\}.$$

We have an inclusion $i : \mathbb{D} \to \mathbb{D}^t$, and functors $G : \mathbb{D}^t \to \mathbb{A}$ such that $Gi = F$ correspond exactly to cocones under $F$.

Let $G : \mathbb{D}^t \to \mathbb{A}$ be such a functor; we need to construct a map $z_G : F^a s \to Gt$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "Y"] \arrow[d, "i"'] \arrow[dr, "Yi"'] & \mathrm{Psh}\,\mathbb{D} \arrow[d, "(Yi)^*"] \\
\mathbb{D}^t \arrow[r, "Y"] \arrow[dr, "G"'] & \mathrm{Psh}\,\mathbb{D}^t \arrow[d, "G^a"] \\
& \mathbb{A}
\arrow[from=2-1, to=1-2, phantom, "\overset{\eta_{Yi}}{\Longrightarrow}"]
\arrow[from=3-2, to=2-2, phantom, near start, "\overset{\tilde{a}}{\Longrightarrow}"]
\end{tikzcd}$$

and consider the objects $s \in \mathrm{Psh}\,\mathbb{D}$ and $Yt \in \mathrm{Psh}\,\mathbb{D}^t$. Since $t$ is terminal in $\mathbb{D}^t$, $Yt$ is terminal in $\mathrm{Psh}\,\mathbb{D}^t$. Define the presheaf

$$X := (Yi)^* s$$

in $\mathrm{Psh}\,\mathbb{D}^t$ (explicitly, this is the presheaf that sends objects $d$ to singletons and $t$ to the empty set). Then we have a unique map

$$X \xrightarrow{!} Yt$$

in $\mathrm{Psh}\,\mathbb{D}^t$. Applying the functor $G^a$ to this map, we obtain

$$G^a X \xrightarrow{G^a!} G^a Yt,$$

which we can compose with the following isomorphisms:

$$F^a s = (Gi)^a s \xrightarrow{\tilde{a}} (G^aYi)^a s \xrightarrow{\hat{a}} G^a(Yi)^* s \xrightarrow{G^a!} G^aYt \xrightarrow{\tilde{a}^{-1}} Gt$$

giving us a map $F^a s \to Gt$, and this is how we define $F^a s \xrightarrow{z_G} Gt$.

For (3), we need to show this is a map of cocones, which is to say that the diagram

$$\begin{tikzcd}
Fd \arrow[r, equal] \arrow[d, "\tilde{a}"'] & Gid \arrow[dd, "G!"] \\
F^a Yd \arrow[d, "F^a \upsilon_d"'] & \\
F^a s \arrow[r, "z_G"'] & Gt
\end{tikzcd}$$

commutes. Writing out the definition of $z_G$, we can fill this diagram with two equalities and two natural transformations:

$$\begin{tikzcd}
Fd \arrow[r, equal] \arrow[d, "\tilde{a}"'] & Gid \arrow[r, "\tilde{a}"] \arrow[d, "\tilde{a}"'] & G^a Yid \arrow[d, "\tilde{a}"'] \\
F^a Yd \arrow[r, equal] \arrow[d, "F^a \upsilon_d"'] & (Gi)^a Yd \arrow[r, "\tilde{a}"'] \arrow[d, "\upsilon_d"'] & (G^a Yi)^a Yd \arrow[d, "\upsilon_d"'] \\
F^a s \arrow[r, equal] & (Gi)^a s \arrow[r, "\tilde{a}"'] & (G^a Yi)^a s
\end{tikzcd}$$

as well as two naturality squares, pseudoalgebra coherence (5) and the image of a unique map into $Yt$ under $G^a$:

$$\begin{tikzcd}
G^a Yid \arrow[d, "\tilde{a}"'] \arrow[drr, equal] & & & \\
(G^a Yi)^a Yd \arrow[r, "\hat{a}"'] \arrow[d, "\upsilon_d"'] & G^a (Yi)^* Yd \arrow[r, "\eta^{-1}"'] \arrow[d, "\upsilon_d"'] & G^a Yid \arrow[r, "\tilde{a}^{-1}"] \arrow[d, "G^a Y!"] & Gid \arrow[d, "G!"] \\
(G^a Yi)^a s \arrow[r, "\hat{a}"'] & G^a (Yi)^* s \arrow[r, "G^a !"'] & G^a Yt \arrow[r, "\tilde{a}^{-1}"'] & Gt
\end{tikzcd}$$

Hence $F^a s \xrightarrow{z_G} Gt$ is a map of cocones.

This gives us that $F^a s$ is a weak colimit for the diagram $F$; it remains to do (4) and show uniqueness. Let $G : \mathbb{D}^t \to \mathbb{A}$ correspond to a cocone under $F$, and let $g : F^a s \to Gt$ be a map of cocones; we want to show that $g = z_G$. Define $H : \mathbb{D}^t \to \operatorname{Psh} \mathbb{D}$ as follows:

$$\begin{aligned}
H : \mathbb{D}^t &\to \operatorname{Psh} \mathbb{D} \\
id &\mapsto Yd \\
t &\mapsto s \\
(id \xrightarrow{!} t) &\mapsto (Yd \xrightarrow{\upsilon_d} s),
\end{aligned}$$

noting that $Hi = Y_{\mathbb{D}}$. Then we can define a 2-cell $\beta : F^a H \implies G$ with components

$$\begin{aligned}
\beta_{id} &: F^a Hid = F^a Yd \xrightarrow{\tilde{a}^{-1}} Fd = Gid, \\
\beta_t &: F^a Ht = F^a s \xrightarrow{g} Gt;
\end{aligned}$$

the only nontrivial naturality condition to check is

$$\begin{tikzcd}
F^aHid \arrow[r, "\beta_{id}"] \arrow[d, "F^aH!"'] & Gid \arrow[d, "G!"] \\
F^aHt \arrow[r, "\beta_t"'] & Gt
\end{tikzcd}$$

and this expands to

$$\begin{tikzcd}
F^aYd \arrow[r, "\tilde{a}^{-1}"] \arrow[d, "F^a\upsilon_d"'] & Fd \arrow[r, equal] & Gid \arrow[d, "G!"] \\
F^as \arrow[rr, "g"'] & & Gt
\end{tikzcd}$$

which is precisely the condition that $g$ is a map of cocones. Now, writing out the definition of $z_G$ we obtain

$$F^as = (Gi)^as \xrightarrow{\tilde{a}} (G^aYi)^as \xrightarrow{\hat{a}} G^a(Yi)^*s \xrightarrow{G^a!} G^aYt \xrightarrow{\tilde{a}^{-1}} Gt$$

We use three naturality squares and pseudoalgebra coherence (5):

$$\begin{tikzcd}
F^aH^*(Yi)^*s \arrow[r, "F^aH^*!"] \arrow[d, "\hat{a}^{-1}"'] & F^aH^*Yt \arrow[d, "\hat{a}^{-1}"'] \arrow[dr, "\eta^{-1}"] & \\
(F^aH)^a(Yi)^*s \arrow[r, "(F^aH)^a!"] \arrow[d, "\beta"'] & (F^aH)^aYt \arrow[r, "\tilde{a}^{-1}"'] \arrow[d, "\beta"'] & F^aHt \arrow[d, "\beta_t=g"] \\
G^a(Yi)^*s \arrow[r, "G^a!"'] & G^aYt \arrow[r, "\tilde{a}^{-1}"'] & Gt
\end{tikzcd}$$

along with four naturality squares and pseudoalgebra coherence (3):

$$\begin{tikzcd}
F^a(Hi)^*s \arrow[r, "\eta"] \arrow[d, "\hat{a}^{-1}"'] & F^a(H^*Yi)^*s \arrow[r, "\mu"] \arrow[d, "\hat{a}^{-1}"] & F^aH^*(Yi)^*s \arrow[dd, "\hat{a}^{-1}"] \\
(F^aHi)^as \arrow[r, "\eta"'] \arrow[d, equal] & (F^aH^*Yi)^as \arrow[d, "\hat{a}^{-1}"] & \\
(F^aHi)^as \arrow[r, "\tilde{a}"'] \arrow[d, "\beta"'] & ((F^aH)^aYi)^as \arrow[r, "\hat{a}"'] \arrow[d, "\beta"] & (F^aH)^a(Yi)^*s \arrow[d, "\beta"] \\
(Gi)^as \arrow[r, "\tilde{a}"'] & (G^aYi)^as \arrow[r, "\hat{a}"'] & G^a(Yi)^*s
\end{tikzcd}$$

and finally two equalities and pseudoalgebra coherence (4):

$$\begin{tikzcd}
F^as \arrow[r, "\theta^{-1}"] \arrow[ddr, equal] & F^aY^*s \arrow[r, equal] \arrow[d, "\hat{a}^{-1}"] & F^a(Hi)^*s \arrow[d, "\hat{a}^{-1}"] \\
& (F^aY)^as \arrow[r, equal] \arrow[d, "\tilde{a}^{-1}"] & (F^aHi)^as \arrow[d, "\beta"] \\
& F^as \arrow[r, equal] & (Gi)^as
\end{tikzcd}$$

which allows us to rewrite the composite defining $z_G$ as:

$$F^as \xrightarrow{\theta^{-1}} F^aY^*s = F^a(Hi)^*s \xrightarrow{\eta} F^a(H^*Yi)^*s \xrightarrow{\mu} F^aH^*(Yi)^*s \xrightarrow{F^aH^*Y!} F^aH^*Yt \xrightarrow{\eta^{-1}} F^aHt \xrightarrow{g} Gt.$$

Now this is of the form

$$F^a s \xrightarrow{F^a(\ldots)} F^a s \xrightarrow{g} Gt$$

for some map $s \to s$. But since $s$ is terminal in $\mathrm{Psh}\,\mathbb{D}$, the only such map is $1_s$. Hence by functoriality we have

$$z_G = g \circ F^a(1_s) = g \circ 1_{F^a s} = g.$$

So indeed the map of cocones $F^a s \to Gt$ is unique, which implies

$$F^a s = F^a \operatorname{colim} Y_{\mathbb{D}} \cong \operatorname{colim} F.$$

Hence every presheaf pseudoalgebra $\mathbb{A}$ is cocomplete. □

## References

[FGHW18] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures. *Selecta Mathematica*, 24(3):2791–2830, 2018.

School of Mathematics, University of Leeds
*Email address*: `mmawsl@leeds.ac.uk`