PSEUDOALGEBRAS FOR THE PRESHEAF RELATIVE PSEUDOMONAD ARE COCOMPLETE

ANDREW SLATTERY

1. Relative pseudomonads and their pseudoalgebras

Definition 1.1. Let \mathbb{C}, \mathbb{D} be bicategories and $J : \mathbb{D} \to \mathbb{C}$ a pseudofunctor. A relative pseudomonad $(T, i, ^*; \eta, \mu, \theta)$ along J comprises:

- for every object $X \in \mathbb{D}$ an object $TX \in \mathbb{D}$ and map $i_X : JX \to TX$ in \mathbb{C} , and
- a family of functors $(-)_{X,Y}^* : \mathbb{C}(JX,TY) \to \mathbb{C}(TX,TY)$ for $X,Y \in \mathbb{D}$,

along with three families of invertible 2-cells:

- $\eta_f : f \to f^*i \text{ for } f : JX \to TY,$
- $\mu_{f,g}: (f^*g)^* \to f^*g^*$ for $f: JX \to TY, g: JW \to TX$, and
- $\theta_X : i_X^* \to 1_{TX} \text{ for } X \in \mathbb{D},$

such that the following two coherence diagrams commute:

(i) for $f: JX \to TY$, $g: JW \to TX$, $h: JV \to TW$,

Definition 1.2. Let T be a relative pseudomonad along $J : \mathbb{D} \to \mathbb{C}$. A pseudoalgebra $(A, {}^{a}; \tilde{a}, \hat{a})$ comprises:

- an object $A \in \mathbb{C}$,
- a family of functors $(-)_X^a : \mathbb{C}(JX, A) \to \mathbb{C}(TX, A)$ for $X \in \mathbb{D}$,

along with two families of invertible 2-cells

- $\tilde{a}_f: f \to f^a i \text{ for } f: JX \to A,$
- $\hat{a}_{f,g}: (f^a g)^a \to f^a g^*$ for $f: JX \to A, g: JW \to TX,$

such that the following two coherence diagrams commute:

Date: September 29, 2023.

(i) for
$$f: JX \to A, g: JW \to TX, h: JV \to TW,$$

$$\begin{array}{ccc} ((f^ag)^ah)^a & \xrightarrow{\hat{a}_{f^ag,h}} & (f^ag)^ah^* \\ & & & \downarrow \hat{a}_{f,g}h^* \\ (\hat{a}_{f,g}h)^a & & (f^ag^*)h^* \\ & & & \downarrow \ddots \\ & & & & \downarrow \ddots \\ & & & & (f^a(g^*h))^a \xrightarrow{\hat{a}_{f,g^*h}} f^a(g^*h)^* \xrightarrow{f^a\mu_{g,h}} f^a(g^*h^*) \end{array}$$

(ii) for $f: JX \to A$,

Lemma 1.3. Let \mathbb{A} be a pseudoalgebra over the relative pseudomonad T. Then for $f: JX \to A, g: JW \to TX$, the diagram

(5)
$$\begin{array}{c} f^{a}g \xrightarrow{\tilde{a}} (f^{a}g)^{a}i \xrightarrow{\hat{a}} (f^{a}g^{*})i \\ & & \downarrow^{\sim} \\ f^{a}\eta \longrightarrow f^{a}(g^{*}i) \end{array}$$

 $also\ commutes.$

Proof. Since \tilde{a} is a natural isomorphism, the required coherence is equivalent to

$$\begin{array}{cccc} (f^ag)^ai & \stackrel{\tilde{a}}{\longrightarrow} ((f^ag)^ai)^ai & \stackrel{\hat{a}}{\longrightarrow} ((f^ag^*)i)^ai \\ & & & \downarrow \sim \\ & & & & & (f^a(g^*i))^ai \end{array}$$

and so it suffices to show

$$\begin{array}{cccc} (f^ag)^a & \stackrel{\tilde{a}}{\longrightarrow} ((f^ag)^ai)^a & \stackrel{\hat{a}}{\longrightarrow} ((f^ag^*)i)^a \\ & & & & \downarrow \sim \\ & & & & & (f^a(g^*i))^a \end{array}$$

commutes. Using this instance of equation (3)

$$\begin{array}{c|c} ((f^ag)^ai)^a & \stackrel{\hat{a}}{\longrightarrow} ((f^ag^*)i)^a & \stackrel{\sim}{\longrightarrow} (f^a(g^*i))^a \\ & & & \downarrow^{\hat{a}} \\ & & & & \uparrow^a(g^*i)^* \\ & & & \downarrow^{\mu} \\ (f^ag)^ai^* & \stackrel{\hat{a}}{\longrightarrow} (f^ag^*)i^* & \stackrel{\sim}{\longrightarrow} f^a(g^*i^*) \end{array}$$

 $\mathbf{2}$

the clockwise composite becomes

$$\begin{array}{cccc} (f^ag)^a & \stackrel{\tilde{a}}{\longrightarrow} ((f^ag)^a i)^a & \stackrel{\hat{a}}{\longrightarrow} (f^ag)^a i^* & \stackrel{\hat{a}}{\longrightarrow} (f^ag^*)i^* \\ & & \downarrow^{\sim} \\ & & f^a(g^*i^*) \\ & & \downarrow^{\mu^{-1}} \\ & & f^a(g^*i)^* \\ & & \downarrow^{\hat{a}^{-1}} \\ & & (f^a(g^*i))^a \end{array}$$

and now using equation (4) we can replace the first two maps in the clockwise composite to obtain:

$$\begin{array}{cccc} (f^ag)^a & \xrightarrow{\sim} & (f^ag)^a 1 \xrightarrow{\theta^{-1}} & (f^ag)^a i^* & \xrightarrow{\hat{a}} & (f^ag^*)i^* \\ & & & \downarrow^{\sim} \\ & & & f^a(g^*i^*) \\ & & & \downarrow^{\mu^{-1}} \\ & & & f^a(g^*i)^* \\ & & & \downarrow^{\hat{a}^{-1}} \\ & & & (f^a(g^*i))^a \end{array}$$

which we fill in with four naturality squares, a bicategory coherences and relative pseudomonad coherence (2):

Here the anticlockwise composite is equal to $(f^a \eta_g)^a$, as desired.

2. The presheaf relative pseudomonad

Write Cat for the 2-category of small categories, and write CAT for the 2-category of locally-small categories. Since the category of presheaves on a small

category is in general only locally small, it is natural to ask whether the presheaf construction

$$X \mapsto \operatorname{Psh} X := [X^{op}, \operatorname{Set}]$$

can be given the structure of a relative pseudomonad along the inclusion 2-functor $J: Cat \to CAT$.

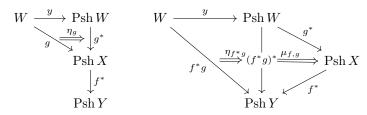
This is shown in [FGHW18] via the construction of a relative pseudoadjunction; the structure of a relative pseudomonad is given to Psh as follows:

- for an object $X \in \text{Cat}$ we have $\text{Psh} X \in \text{CAT}$ and unit map $y_X : X \to \text{Psh} X$ given by the Yoneda embedding,
- for $X, Y \in \text{Cat}$ and a functor $f : X \to \text{Psh} Y$, the extension $f^* : \text{Psh} X \to \text{Psh} Y$ is given by the left Kan extension of f along the Yoneda embedding

$$\begin{array}{ccc} X & \stackrel{y}{\longrightarrow} \operatorname{Psh} X \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ &$$

which also defines the 2-cells $\eta_f : f \to f^* y$ (note that since the Yoneda embedding is fully faithful the maps η_f are invertible, as required),

• for $f: JX \to TY$ and $g: JW \to TX$, the 2-cell $\mu_{f,g}: (f^*g)^* \to f^*g^*$ is uniquely determined by the universal property of the left Kan extension:



• for $X \in \text{Cat}$, the 2-cell $\theta_X : y_X^* \to 1$ is also uniquely determined by the universal property of the left Kan extension:

$$\begin{array}{cccc} X & \stackrel{y}{\longrightarrow} \operatorname{Psh} X & & X & \stackrel{y}{\longrightarrow} \operatorname{Psh} X \\ & & & & & \\ & & & & \\ y & \stackrel{1}{\longrightarrow} \downarrow^{1} & & & \\ & & & & y & \stackrel{\eta_{y}}{\longrightarrow} i^{*}_{*} & \stackrel{\theta_{X}}{\longrightarrow} \downarrow^{1} \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \end{array}$$

The presheaf construction being the free cocompletion, we expect that pseudoalgebras over Psh should be exactly the locally-small cocomplete categories.

Proposition 2.1. Let $(\mathbb{A}, {}^{a}; \tilde{a}, \hat{a})$ be a pseudoalgebra for the presheaf relative pseudomonad. Then $\mathbb{A} \in CAT$ is cocomplete, and the colimit of the diagram $F : \mathbb{D} \to \mathbb{A}$ for $\mathbb{D} \in Cat$ is given by

$$\operatorname{colim} F \cong F^a \operatorname{colim} Y_{\mathbb{D}}.$$

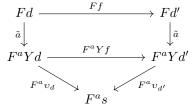
Proof. Define $s := \operatorname{colim} Y_{\mathbb{D}} \in \operatorname{Psh} \mathbb{D}$ (s standing for 'singleton', since it is the terminal presheaf on \mathbb{D} sending every d to a singleton set). Let \mathbb{A} and $F : \mathbb{D} \to \mathbb{A}$ be as hypothesised. Our proof proceeds as follows:

- (1) Show that $F^a s$ is the apex of a cocone under F.
- (2) Given any cocone under F with apex Gt, construct a map $z_G: F^a s \to Gt$.
- (3) Show that z_G is a map of cocones.
- (4) Show that if $g: F^a s \to Gt$ is a map of cocones parallel to z_G , then $g = z_G$.

For (1), consider the colimit cocone under $Y_{\mathbb{D}}$ whose legs $v_d : Yd \to \operatorname{colim} Y$ are the colimit inclusions. Now for $d \in \mathbb{D}$ we can construct the composite

$$Fd \xrightarrow{\tilde{a}} F^a Yd \xrightarrow{F^a v_d} F^a s.$$

Since the following diagram comprising a naturality square and the image under F^a of a cocone:



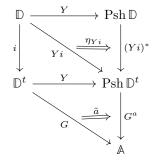
commutes for all $f: d \to d'$, these form the legs of the required cocone under F with apex $F^a s$.

For (2), we begin by characterising cocones under F. Define $\mathbb{D}^t \in \text{Cat}$ to be the category formed by freely adding a terminal object to \mathbb{D} ; that is,

$$ob \mathbb{D}^{t} = ob \mathbb{D} \sqcup \{t\},$$
$$mor \mathbb{D}^{t} = mor \mathbb{D} \sqcup \{1_{t}\} \sqcup \{d \xrightarrow{!_{d}} t : d \in ob \mathbb{D}\}.$$

We have an inclusion $i : \mathbb{D} \to \mathbb{D}^t$, and functors $G : \mathbb{D}^t \to \mathbb{A}$ such that Gi = F correspond exactly to cocones under F.

Let $G : \mathbb{D}^t \to \mathbb{A}$ be such a functor; we need to construct a map $z_G : F^a s \to Gt$. Consider the following commutative diagram:



and consider the objects $s \in \operatorname{Psh} \mathbb{D}$ and $Yt \in \operatorname{Psh} \mathbb{D}^t$. Since t is terminal in \mathbb{D}^t , Yt is terminal in $\operatorname{Psh} \mathbb{D}^t$. Define the presheaf

$$X := (Yi)^* s$$

in $\operatorname{Psh} \mathbb{D}^t$ (explicitly, this is the presheaf that sends objects d to singletons and t to the empty set). Then we have a unique map

$$X \xrightarrow{!} Yt$$

in $\operatorname{Psh} \mathbb{D}^t$. Applying the functor G^a to this map, we obtain

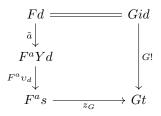
$$G^a X \xrightarrow{G^a !} G^a Y t,$$

which we can compose with the following isomorphisms:

$$F^{a}s = (Gi)^{a}s \xrightarrow{\tilde{a}} (G^{a}Yi)^{a}s \xrightarrow{\hat{a}} G^{a}(Yi)^{*}s \xrightarrow{G^{a}!} G^{a}Yt \xrightarrow{\tilde{a}^{-1}} Gt$$

giving us a map $F^a s \to Gt$, and this is how we define $F^a s \xrightarrow{z_G} Gt$.

For (3), we need to show this is a map of cocones, which is to say that the diagram



commutes. Writing out the definition of z_G , we can fill this diagram with two equalities and two natural transformations:

$$\begin{array}{cccc} Fd & & & & & Gid & & \stackrel{\tilde{a}}{\longrightarrow} & G^{a}Yid \\ & & & & & & \\ \tilde{a} & & & & & \\ F^{a}Yd & & & & & & \\ F^{a}v_{d} & & & & & \\ F^{a}v_{d} & & & & v_{d} \\ F^{a}s & & & & & & \\ F^{a}s & & & & & & \\ \end{array}$$

as well as two naturality squares, pseudoalgebra coherence (5) and the image of a unique map into Yt under G^a :

$$\begin{array}{c} G^{a}Yid \\ \tilde{a} \\ (G^{a}Yi)^{a}Yd \xrightarrow{\hat{a}} G^{a}(Yi)^{*}Yd \xrightarrow{\eta^{-1}} G^{a}Yid \xrightarrow{\tilde{a}^{-1}} Gid \\ v_{d} \\ v_{d} \\ (G^{a}Yi)^{a}s \xrightarrow{\hat{a}} G^{a}(Yi)^{*}s \xrightarrow{G^{a}!} G^{a}Yt \xrightarrow{\tilde{a}^{-1}} Gt \end{array}$$

Hence $F^a s \xrightarrow{z_G} Gt$ is a map of cocones.

This gives us that $F^a s$ is a weak colimit for the diagram F; it remains to do (4) and show uniqueness. Let $G : \mathbb{D}^t \to \mathbb{A}$ correspond to a cocone under F, and let $g : F^a s \to Gt$ be a map of cocones; we want to show that $g = z_G$. Define $H : \mathbb{D}^t \to \operatorname{Psh} \mathbb{D}$ as follows:

$$\begin{aligned} H: \mathbb{D}^t &\to \operatorname{Psh} \mathbb{D} \\ id &\mapsto Yd \\ t &\mapsto s \\ (id \xrightarrow{!} t) &\mapsto (Yd \xrightarrow{v_d} s), \end{aligned}$$

noting that $Hi = Y_{\mathbb{D}}$. Then we can define a 2-cell $\beta : F^a H \implies G$ with components

$$\beta_{id}: F^a Hid = F^a Yd \xrightarrow{\tilde{a}^{-1}} Fd = Gid,$$

$$\beta_t: F^a Ht = F^a s \xrightarrow{g} Gt;$$

6

the only nontrivial naturality condition to check is

and this expands to

which is precisely the condition that g is a map of cocones. Now, writing out the definition of z_G we obtain

$$F^{a}s = (Gi)^{a}s \xrightarrow{\tilde{a}} (G^{a}Yi)^{a}s \xrightarrow{\hat{a}} G^{a}(Yi)^{*}s \xrightarrow{G^{a}!} G^{a}Yt \xrightarrow{\tilde{a}^{-1}} Gt$$

We use three naturality squares and pseudoalgebra coherence (5):

along with four naturality squares and pseudoalgebra coherence (3):

and finally two equalities and pseudoalgebra coherence (4):

$$F^{a}s \xrightarrow{\theta^{-1}} F^{a}Y^{*}s = F^{a}(Hi)^{*}s$$

$$\downarrow^{\hat{a}^{-1}} \qquad \downarrow^{\hat{a}^{-1}}$$

$$(F^{a}Y)^{a}s = (F^{a}Hi)^{a}s$$

$$\downarrow^{\tilde{a}^{-1}} \qquad \downarrow^{\beta}$$

$$F^{a}s = (Gi)^{a}s$$

which allows us to rewrite the composite defining z_G as:

$$F^{a}s \xrightarrow{\theta^{-1}} F^{a}Y^{*}s = F^{a}(Hi)^{*}s \xrightarrow{\eta} F^{a}(H^{*}Yi)^{*}s \xrightarrow{\mu} F^{a}H^{*}(Yi)^{*}s \xrightarrow{F^{a}H^{*}Y!} F^{a}H^{*}Yt \xrightarrow{\eta^{-1}} F^{a}Ht \xrightarrow{g} Gt$$

Now this is of the form

$$F^as \xrightarrow{F^a(\ldots)} F^as \xrightarrow{g} Gt$$

for some map $s \to s$. But since s is terminal in $Psh \mathbb{D}$, the only such map is 1_s . Hence by functoriality we have

$$z_G = g \circ F^a(1_s) = g \circ 1_{F^a s} = g.$$

So indeed the map of cocones $F^a s \to Gt$ is unique, which implies

$$F^a s = F^a \operatorname{colim} Y_{\mathbb{D}} \cong \operatorname{colim} F.$$

Hence every presheaf pseudoalgebra $\mathbb A$ is cocomplete.

References

[FGHW18] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures. Selecta Mathematica, 24(3):2791–2830, 2018.

SCHOOL OF MATHEMATICS, UNIVERSITY OF LEEDS *Email address*: mmawsl@leeds.ac.uk

8